Density Evolution Analysis of LDPC codes with Different Receivers on Impulsive Noise Channels

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***Abstract*—In conventional communication systems, the additive noise at the receiver is usually assumed to be Gaussian distributed. However, this assumption is not always valid and examples of non-Gaussian distributed noise include power line channels, underwater acoustic channels and man-made interference. Therefore it is important to design a receiver to mitigate the effects of impulsive noise. This paper proposes a new low-complexity receiver design that closely approximates the optimal log-likelihood ratio demapper and performs well when the channel varies from slightly impulsive to very impulsive. Furthermore, we present a density evolution analysis and simulation results of (3, 6) LDPC code on additive impulsive noise channels, with symmetric alpha-stable probability density functions, employing different receivers to overcome the impulsive nature of the channels. Each receiver calculates or approximates the log-likelihood ratios of the received symbols, depending on whether they are optimal or sub-optimal respectively. The threshold signal-to-noise ratios of the LDPC code are derived to determine the start of the waterfall region of the bit-error rate performance and simulation results are presented to validate our density evolution analysis.**

***Index Terms*—LDPC codes, impulse noise, sub-optimal receiver, density evolution**

I. Introduction

Low-density parity-check (LDPC) codes are a class of linear block codes that were first presented by Gallager in 1962 [1] and rediscovered by Mackay and Neal in 1996 [2]. LDPC codes are widely applied to different standards such as WiMAX and DVB-S2 due to their near Shannon limit performance and low complexity decoder. Conventionally, a receiver employs a log-likelihood ratio (LLR) demapper to cooperate with the soft decoding process, which is generally a linear receiver that assumes the noise is Gaussian. However, many communication channels such as power line channels and underwater acoustic channels cause the appearance of impulses in the transmitted signal, which last for a very short duration and have amplitudes much greater than the thermal noise added at the receiver. These impulses result in the overall noise at the receiver having a heavy-tailed distribution, which can be simulated using several different model such as Middleton Class A and B noise models [3] or alpha stable distributions [4]. In this paper, we study the performance of LDPC codes on symmetric alpha stable (SαS) channels since SαS distributions have been shown to accurately model impulsive noise [5].

As a consequence of the Gaussian assumption not being valid in all situations, other receivers should be considered instead of the linear receiver. Optimal receivers assume the probability density function (pdf) of the noise is known, but as we will see in the next section, the pdf of SαS distributions have almost no closed form expressions which leads to a huge computational cost to obtain the LLR values of the received symbols. Hence, suboptimal receivers are more practical and these can be very simple limiting devices, such as the soft limiter and hole puncher, that reduces the effect of impulses [6]. However, these limiting operations may introduce undesired distortion which degrades the performance, hence another type of suboptimal receivers has been proposed. This type of suboptimal receiver aims at approximating the pdf of the noise distributions. The Cauchy receiver has shown good performance for a large range of impulsiveness [7] since it provides a good estimate of the algebraic tails of the heavy-tailed pdfs. However, this is far from optimal and this receiver requires knowledge of the noise statistics. Recently, a new sub-optimal receiver was proposed to achieve near optimal performance requiring only the knowledge of the dispersion of the pdf [8].

In this paper, we propose a new receiver which can outperform the receiver in [8] with lower complexity and also approaches the performance of the optimal receiver. Furthermore, the threshold signal-to-noise ratios (SNR) of a (3,6) LDPC code are derived for different receivers by applying a density evolution analysis on channels with different levels of impulsiveness. Additionally we present simulation results of a very long Mackay's (3,6) LDPC code on the same impulsive noise channels to support our analysis.

This paper is organized as follows: Section II introduces SαS distributions, the system model and a definition of the geometric signal-to-noise ratio. Description of the different types of receivers, including our receiver, and the density evolution analysis to derive the threshold SNRs of the different receivers are presented in Section III. In Section IV simulation results are presented and finally conclusions are given in Section V.

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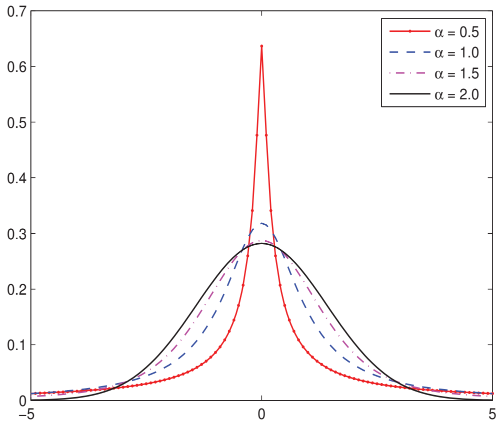

Fig. 1. Standard $S\alpha S$ distributions($\gamma = 1, \delta = 0$)

II. Symmetric Alpha Stable Channels

A. Symmetric Alpha Stable Distributions

The family of alpha stable distributions has been found to be a useful model in many areas, including stock analysis, signal processing, and radar communications. The pdf of symmetric alpha stable distributions is

$$f_\alpha(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(j\delta t - \gamma^\alpha |t|^\alpha) e^{-itx} dt. \tag{1}$$

The $S\alpha S$ distribution in (1) has three parameters, α, γ and δ. 1) α is the characteristic exponent with a range $(0, 2]$ which represents the tail heaviness of the pdf. 2) The parameter γ is called the dispersion which measures the spread of the $S\alpha S$ pdf. 3) δ is the mean or the median of an $S\alpha S$ pdf, depending on the value of α. A general $S\alpha S$ distribution has no closed form expressions except in two cases: $\alpha = 2$ and $\alpha = 1$. When $\alpha = 2$, the distribution is Gaussian and when $\alpha = 1$, we have the Cauchy distribution. As α decreases, the tail of the pdf becomes thicker, which can be observed in Fig. 1, which shows the pdfs of $S\alpha S$ distributions for different α.

B. System Model

The system model comprises a transmitter, additive impulsive noise channel and receiver. At the transmitter, the message is encoded by a LDPC encoder and then mapped to a binary phase shift keying (BPSK) constellation. At the receiver, several different demappers are employed to mitigate the impulses in the received signal and generate LLRs used by the soft decision decoder. The channel output $\boldsymbol{Y}$ is denoted by

$$\boldsymbol{Y} = \boldsymbol{X} + \boldsymbol{N}, \tag{2}$$

where $\boldsymbol{X}$ is a BPSK modulated sequence and the symbols in $\boldsymbol{X}$ belong to $\{-1, 1\}$. $\boldsymbol{N}$ is a sequence of independent $S\alpha S$ distributed noise samples.

C. Geometric Signal-to-Noise Ratio

The second-order moment of $S\alpha S$ random variables is infinite, hence so called fractional lower order moments (FLOM) are useful to characterize the impulsive signals since only moments of order less than α exist.

$$E\{|X|^p\} < \infty, \text{if } 0 \le p < \alpha. \tag{3}$$

where $E\{\cdot\}$ is the expectation operator. However, if $p \ge \alpha$, the associated FLOMs are not defined so zero-order statistics (ZOS) were proposed to characterize the $S\alpha S$ process [9]. The power, as a second order moment, is widely used as a measure of the signal strength, but in the case of $S\alpha S$ distributions ($\alpha < 2$), the second order power is infinite. Thus the logarithmic-order moments $E\{\log |X|\}$ are employed to define the power, since $E\{\log |X|\} < \infty$. Then the geometric power of X is defined as

$$S_0(X) = e^{E\{\log |X|\}} \tag{4}$$

If we apply (4) to an $S\alpha S$ process, a closed form expression for geometric power can be expressed as

$$S_0 = \frac{(C_g)^{1/\alpha}\gamma}{C_g}, \tag{5}$$

where $C_g \approx 1.78$ is the exponential of the Euler constant. Hence, the geometric SNR, $\mathrm{SNR_G}$, of a signal with amplitude A is given as [9]

$$\mathrm{SNR_G} = \frac{1}{2C_g}\left(\frac{A}{S_0}\right)^2, \tag{6}$$

where $1/(2C_g)$ is a normalization constant which ensures the geometric signal-to-noise ratio can still be applied when $\alpha = 2$. However, it is still convenient to evaluate the bit-error rate performance as a function of $\frac{E_b}{N_0}$, which we can define for BPSK modulation in terms of the geometric SNR and code rate, R, as

$$\frac{E_b}{N_0} = \frac{\mathrm{SNR_G}}{2R} = \frac{1}{4RC_g}\left(\frac{A}{S_0}\right)^2. \tag{7}$$

III. Receivers and Density Evolution Analysis

A. Optimal and Suboptimal Receivers

Let x_j be the j-th symbols of $\boldsymbol{X}$ and y_j be the j-th symbols of $\boldsymbol{Y}$. The LLR demapper of the optimum receiver of a memoryless binary symmetric channel is given as

$$L_j = \ln\left(\frac{P(y_j|x_j = 1)}{P(y_j|x_j = -1)}\right) = \ln\left(\frac{f_n(y_j - 1)}{f_n(y_j + 1)}\right), \tag{8}$$

where $f_n(x)$ is the pdf of the noise $\boldsymbol{N}$. However, if we use this optimum LLR to initialize the decoder, the computation cost is high due to the integration in (1). Hence, a suboptimal receiver is necessary to reduce the complexity while at the same time achieve a performance approaching the optimal receiver.

Now we discuss some of the most recent suboptimal receivers presented in the literature. As mentioned previously, the pdf of an $S\alpha S$ distributions has a closed form expression when $\alpha = 1$. The pdf of this Cauchy distribution is suitable to model the heavy-tailed distributions for a large range of α since it exhibits algebraic tails.

A much simpler receiver is a variant of the soft limiter called the clipper, which was proposed in [10]. The equation of this LLR demapper is

$$L_j = \begin{cases} py_j, & \text{if } -h/p < y_j < h/p \\ h\text{sign}(y_j), & \text{otherwise.} \end{cases} \quad (9)$$

where p is the amplitude of the signal and h is the clipping level of the impulse. The optimized p's and h's for different α were given in [10] using density evolution.

A receiver to approximate the LLR values of the received symbols was proposed recently that achieves near-optimal performance [8]. It decomposes the LLR demapper into two parts: a linear part and an asymptotic part. The linear part is related to the dispersion γ and the asymptotic part is obtained by approximating the asymptotic expansion of the $S\alpha S$ pdf. The LLR demapper for positive y_j is given as

$$L_j = \min\left(\frac{\sqrt{2}y_j}{\gamma}, \frac{2(\alpha+1)}{y_j}\right). \quad (10)$$

The LLR for negative y_j is obtained by replacing the min operation in (10) by a max operation. However, the linear part still requires an estimation of the dispersion γ of the pdf.

We propose a new demapper to approximate the LLRs, which can achieve a near-optimal performance without knowledge of the pdf or the dispersion γ. For our receiver, the linear part of (10) is replaced with py_j where p is the optimized gradient of the clipper. Hence this receiver only requires knowledge of α and the estimation of parameters of alpha stable noise including α was proposed in [11]. This new LLR demapper is defined as

$$L_j = \begin{cases} \min(py_j, \frac{2(\alpha+1)}{y_j}) & y_j \geq 0 \\ \max(py_j, \frac{2(\alpha+1)}{y_j}) & y_j < 0 \end{cases}. \quad (11)$$

Fig. 2 show plots of the LLR demappers for the optimal and suboptimal receivers discussed in this section when $\alpha = 1.6$ and E_b/N_0 is 2dB. Compared with the Cauchy and clipper receivers, our receiver closely matches the curve of the optimal LLR demapper, which suggests that the performance of our receiver should approach the optimal performance. Compared with LLR approximation receiver, our receiver better fits the linear part of the true LLR. In the next section, we will evaluate the asymptotic performance of the different receivers using density evolution to support this argument.

B. Density Evolution Analysis on $S\alpha S$ channels

A (d_v, d_c) LDPC code is defined by a sparse parity check matrix $\boldsymbol{H}$ and has d_c 1s in each row and d_v 1s in each column. The designed code rate of this regular LDPC code is

$$R_c = 1 - \frac{d_v}{d_c}. \quad (12)$$

In this paper, we use a $(3, 6)$ regular LDPC code with block size 20,000 bits to examine the asymptotic performance of different receivers. There are three tools to analyze the behavior of iterative decoders: Density Evolution (DE), Gaussian Approximation (GA) and the extrinsic information transfer (EXIT) chart. However, GA and EXIT assume the channel is AWGN or the extrinsic information transferred between two decoders is Gaussian distributed, which is no longer correct for the case of additive $S\alpha S$ noise channels.

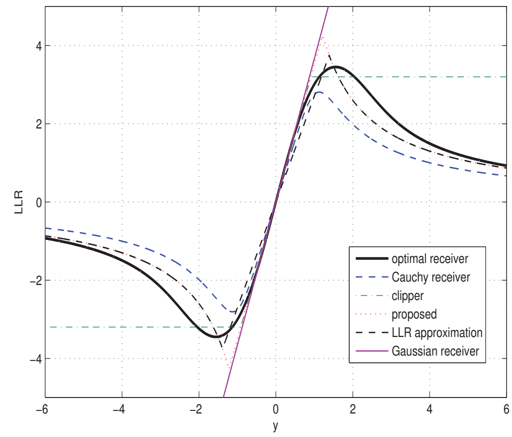

Fig. 2. LLR demapper ($\alpha = 1.6$, $E_b/N_0 = 2$ dB)

The density evolution technique assumes the block length is infinite and it allows us to determine the threshold dispersion, γ^*, of a specific LDPC code ensemble, which is defined as

$$\gamma^* = \sup\left\{\gamma : \lim_{l\to 0}\int_{-\infty}^{0} p_v^{(l)}(x)dx = 0\right\}, \quad (13)$$

where $p_v^{(l)}$ is the pdf of the outgoing message for a variable node in the l-th iteration. For $\gamma < \gamma^*$, the decision error converges to zero and the corresponding value of γ^* is guaranteed to be a lower bound for the dispersion, indicating where the waterfall region of the BER curve begins.

Here we give the procedure of our density evolution for regular LDPC codes which was proposed in [12]. First, the initial LLR density $p_v^{(0)}$ is calculated. In our case, $p_v^{(0)}$ from the $S\alpha S$ channels has no analytical expression, hence Monte-Carlo simulation and a histogram method are employed to find the distribution of the initial LLRs. Then a two-step procedure which contains the density evolution of the check nodes and variable nodes is performed iteratively. The check node update is denoted as

$$u_j^{(l)} = 2\tanh^{-1}\left(\prod_{i=1}^{d_c-1}\tanh\left(\frac{v_i^{(l-1)}}{2}\right)\right), \quad (14)$$

where $u_j^{(l)}$ and $v_i^{(l)}$ are the message of j-th check node and the message of i-th variable node in the l-th iteration respectively. To find the density evolution of this step, a G-density is defined as the pdf of $g(y)$ [13], where

$$g(y) = \left(\text{sign}\left(l(y)\right), \ln\coth\left(\frac{|l(y)|}{2}\right)\right). \quad (15)$$

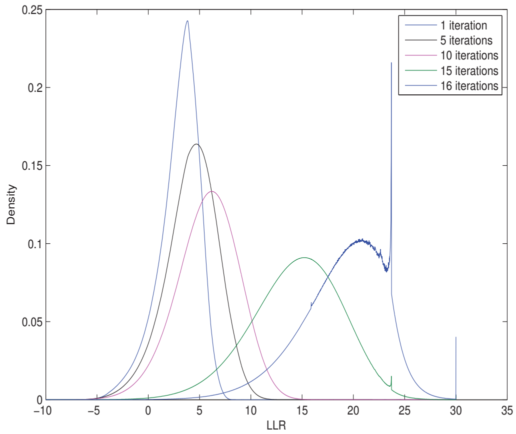

Fig. 3. Density evolution at variable node with the optimal receiver, when $\alpha = 1.8$ and $E_b/N_0 = 2$ dB.

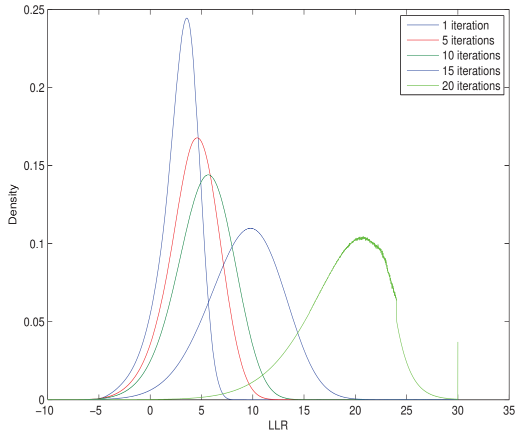

Fig. 4. Density evolution at variable node with our proposed receiver, when $\alpha = 1.8$ and $E_b/N_0 = 2$ dB.

Here $l(y)$ represents LLR in (8) and (14) can be written as

$$u_j^{(l)} = g^{-1}\left(\sum_{i=1}^{d_c-1} g(v_i^{(l-1)})\right). \tag{16}$$

With this notation, the pdf of $u_j^{(l)}$ can be simply expressed as the convolution of $d_c - 1$ G-densities. The density evolution of a check node is denoted as

$$p_u^{(l)} = \left(p_v^{(l-1)}\right)^{\boxtimes(d_c-1)}, \tag{17}$$

where $p_v^{(l)}$ is the pdf of each $v_i^{(l)}$ and $p_u^{(l)}$ is the pdf of each $u_j^{(l)}$ in the l-th iteration. The notation $\boxtimes$ denotes the transformation from $p_v^{(l-1)}$ to the G-density domain, the convolution in the G-density domain and finally the conversion back to $p_u^{(l)}$.

The variable to check message is

$$v_i^{(l)} = v_i^{(0)} + \sum_{j=1}^{d_v-1} u_j^{(l)}, \tag{18}$$

where $v_i^{(0)}$ is the initial LLR of the received. Hence the density evolution of this step is the convolution of these densities. Since the incoming messages are independent, we have

$$p_v^{(l)} = p_v^{(0)} \otimes \left(p_u^{(l)}\right)^{\otimes(d_v-1)}, \tag{19}$$

where $\otimes$ denotes the convolution operation. These convolutions can be efficiently calculated by employing fast Fourier transforms (FFT),

$$p_v^{(l)} = F^{-1}\left(F\left[p_v^{(0)}\right] \cdot \left(F\left[p_u^{(l)}\right]\right)^{d_v-1}\right). \tag{20}$$

This two-stage process is repeated a number of times to calculate the threshold dispersion. To determine the threshold SNR we substitute γ^* into (5) to calculate S_0 and obtain E_b/N_0 from (7).

The threshold E_b/N_0 in dB for different receivers are given in table I. As shown in this table, the threshold SNRs for the LLR approximation receiver and our proposed receiver are very close to optimal for all ranges of α. In addition, our receiver outperforms the LLR approximation receiver for some values of α. When $\alpha = 1$ the Cauchy receiver becomes an optimal receiver and naturally achieves the best performance when the noise has a Cauchy pdf, but it also performs well for a large range of α. However, the Cauchy receiver suffers from significant degradation in performance when the channel is only slightly impulsive, namely when $\alpha = 1.8$. The clipper is also attractive since its performance is near optimal when $\alpha = 1.8$ and its threshold SNR approaches the threshold SNR for the optimum receiver when the channel become less impulsive. This shows that the clipper receiver is more suitable for slightly impulsive channels due to its performance in these conditions and its low complexity.

TABLE I
THE THRESHOLD SNRS IN DB FOR THE DIFFERENT RECEIVERS

	optimal	LLR appro.	proposed	Cauchy	Clipper
$\alpha = 1.8$	1.54	1.64	1.63	1.90	1.65
$\alpha = 1.6$	1.88	1.98	1.98	2.08	2.05
$\alpha = 1.2$	2.72	2.79	2.78	2.76	3.55
$\alpha = 1.0$	3.31	3.38	3.36	3.31	4.80

Additionally, we have plotted the densities of the variable node output for different receivers when $\alpha = 1.8$ and E_b/N_0 is 2 dB in Fig. 3- 5. According to (13), the total probability of error is the integration of $p_v^{(l)}$ corresponds to LLR from $-\infty$ to 0. As shown in Fig. 3 - 5, the area of the pdf for which the LLR ranges from $-\infty$ to 0 becomes smaller as the number of iterations increases. Furthermore, the Cauchy receiver requires 50 iterations to make the decision error converge to zero while the optimal receiver and our receiver only need 16 and 20

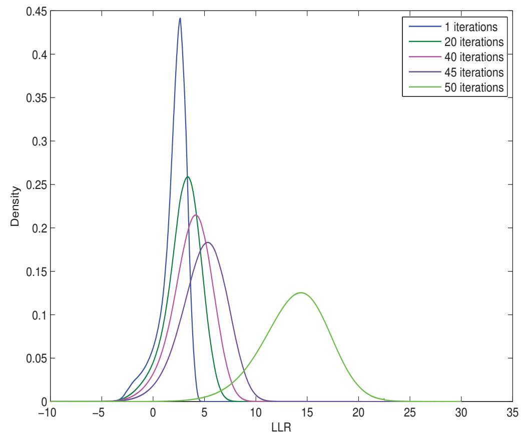

Fig. 5. Density evolution at variable node with Cauchy receiver when $\alpha = 1.8$ and $E_b/N_0 = 2$ dB

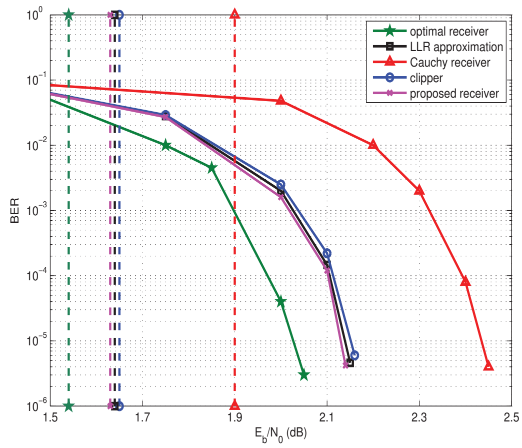

Fig. 6. LDPC codes ($n_b = 20000$ bits) on the $S\alpha S$ channel with $\alpha = 1.8$

iterations respectively. This result matches the thresholds we obtained in table I for the Cauchy, optimal and proposed LLR approximation receivers, which are 1.90 dB, 1.54 dB and 1.63 dB respectively.

IV. Simulation Results

To examine the performance of different receivers and validate the thresholds, simulation results of Mackay's $(3, 6)$ LDPC code with block size $n_b = 20000$ bits and $R = 0.5$ are presented. The maximum number of iterations is 20. We present the BER performance of different receivers for $\alpha = 1.8$ and $\alpha = 1$ which represent slightly impulsive and very impulsive channels respectively. In each case, the sub-optimal receivers are compared with the optimal receiver, which has the best performance since it knows the pdf of the noise and the dispersion. In Fig. 6, when $\alpha = 1.8$ the clipper, LLR approximation receiver and our receiver perform similarly and are about 0.1 dB away from the optimal receiver, while the Cauchy receiver is about 0.4 dB worse than the

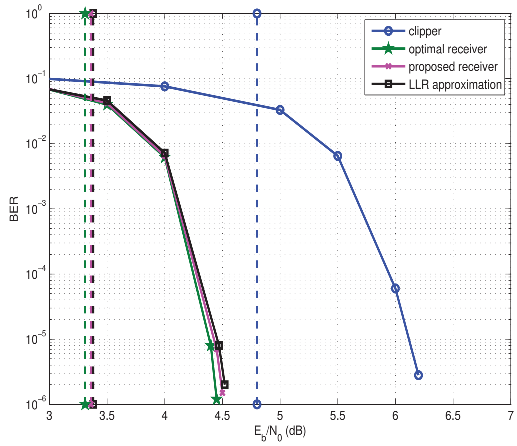

Fig. 7. LDPC codes ($n_b = 20000$ bits) on the $S\alpha S$ channel with $\alpha = 1$

optimal receiver. In Fig. 7, $\alpha = 1$ the Cauchy receiver becomes the optimal receiver since the channel now has a Cauchy pdf. Our receiver matches the performance of the optimal receiver and the LLR approximation receiver is only 0.1 dB worse. However, the performance of the clipper is 1.7 dB worse than the optimal receiver.

Superimposed on Figs. 7 and 6 are the threshold SNRs for the different receivers derived from the density evolution analysis, denoted by dashed vertical lines. We observe that the threshold SNRs match closely with the start of the waterfall regions for the BER plots of each receiver, thus showing that the simulation results validate our analysis.

V. Conclusion

In this paper, we have investigated the performance of Mackay's (3 6) LDPC code on additive impulsive noise channels with different receivers designed to overcome impulses by calculating or approximating the LLRs of the received symbols. We have proposed a low-complexity sub-optimal receiver that produces a very good approximation of the LLRs but does not require knowledge of the pdf of the impulsive noise or its dispersion. An analysis of the LDPC code was presented by applying density evolution to each receiver on different impulsive noise channels to derive the threshold SNRs that indicate the start of the waterfall region of the BER plots. Furthermore, simulation results of the LDPC code on additive impulsive noise channels with the different receivers were also presented to validate the density evolution analysis.

The density evolution analysis and the simulation results show that our receiver achieves near-optimal performance for a large range of α, requiring the least amount of knowledge of the statistics of the channel. We also found that the clipper is most suitable when the channel is only slightly impulsive due to its simplicity and near optimal performance in these conditions, whereas the Cauchy receiver is most suitable when α approaches 1. The LLR approximation receiver also performs well over a large range of α and in some case matches the performance of our receiver, but there are values of α where our receiver has a slight advantage in performance due to the better approximation of the LLRs of the received symbols. We can conclude that our receiver is a good choice for channels with light or heavy impulsiveness, achieving a very good performance while maintaining a low complexity since it does not require knowledge of the pdf of the noise or its dispersion.

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