Concatenated Reed-Solomon/Spatially Coupled LDPC Codes

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***Abstract*—Spatially coupled (SC) low-density parity-check (LDPC) codes can achieve capacity approaching belief propagation (BP) decoding performance with low message recovery latency when using the sliding window decoding (SWD). The SWD is suitable for data stream transmission but cannot always correct all errors in the targeted symbol set. A further decoding stage would be desirable to eliminate the remaining errors from SWD. This paper proposes a novel concatenated coding scheme where the Reed-Solomon (RS) codes and the SC-LDPC codes are chosen as the outer code and the inner code, respectively, namely the RS-SC-LDPC codes. Consequently, the remaining errors from SWD can be corrected by the outer RS codes. In the concatenated coding scheme, the inner SC-LDPC code is designed such that it has a large girth and locally systematic encoding property. Our simulation results show the RS-SC-LDPC codes can achieve a high decoding performance, while the error floor of the inner code can be removed.**

***Index Terms*—Concatenated codes, SC-LDPC codes, RS codes**

I. Introduction

Spatially coupled low-density parity-check (SC-LDPC) codes [1] were first introduced as LDPC convolutional codes [2]. Compared to LDPC block codes, SC-LDPC codes inherit the advantages of regular [3] and irregular LDPC codes [4], i.e., linear growth of minimum distance with block length and capacity-approaching belief propagation (BP) decoding thresholds. SC-LDPC codes can usually be constructed based on protographs [5] [6]. They can be constructed by first coupling L disjoint block protographs and further applying lifting to the coupled protograph to yield the Tanner graph [7] of the code. In its parity-check matrix, the non-zero entries locate in a diagonal band. Consequently, the sliding window decoding (SWD) [8] can be utilized to decode the SC-LDPC codes resulting in a low message recovery latency.

However, the SWD often yields burst errors within the targeted symbol set [9]. It is desirable to concatenate the SC-LDPC code with an outer code to further eliminate the burst errors. Reed-Solomon (RS) codes are good candidates due to their strong burst error correction property. Concatenated LDPC codes have been widely adopted in several standards, e.g., in China Mobile Multimedia Broadcasting (CMBB) and Digital Video Broadcasting-Satellite 2 (DVB-S2), the RS-LDPC code and the Bose-Chaudhuri-Hocquenghem (BCH)-LDPC codes are used, respectively. Mitchell *et al.* proposed BCH-SC-LDPC codes [10] to improve the error floor performance but at the cost of a slight waterfall performance degradation. A stronger outer code would be desirable to further improve the decoding performance by correcting the burst errors of the inner SWD.

In this paper, we propose a novel concatenated coding scheme, namely the RS-SC-LDPC codes, where the RS codes and the SC-LDPC codes are the outer code and the inner code, respectively. The proposed concatenated code can yield a high decoding performance and maintain a low message recovery latency. This is a new coding scheme for data streaming where higher transmission reliability is required. In order to decode RS codes from the inner SWD, the SC-LDPC codes should satisfy a locally systematic encoding property [11]. Ensuring this property, this paper will also introduce the design of the inner SC-LDPC code such that it has a girth of at least eight. This would result in a high decoding performance of the proposed concatenated code. Simulation results indicate it and show that the error floor of the concatenated codes will not appear until the bit error rate (BER) of 10^{-8}.

II. SC-LDPC Codes

A. Construction of SC-LDPC Codes

SC-LDPC codes can be constructed based on protographs [5]. A block protograph is a bipartite graph with n_c check nodes and n_v variable nodes. It can be described using a base matrix $\mathbf{B}$ of size $n_c \times n_v$, where $\mathbf{B}(r, s)$ denotes its row-r column-s entry with $0 \leq r \leq n_c - 1$ and $0 \leq s \leq n_v - 1$. $\mathbf{B}(r, s)$ represents the number of edges connecting check node r and variable node s. Fig. 1(a) shows a block protograph defined by $\mathbf{B} = [3, 3]$, where $n_c = 1$ and $n_v = 2$. To form a coupled protograph, the block protograph of Fig. 1(a) will be replicated, yielding a chain of block protographs over time as shown in Fig. 1(b). For all block protographs, some of their edges from variable nodes at time instant t will be emancipated and connected to the check nodes at time instants between $t + 1$ and $t + \omega$. This edge spreading can be interpreted by decomposing $\mathbf{B}$ into $\omega + 1$ submatrices of the same size, i.e.,

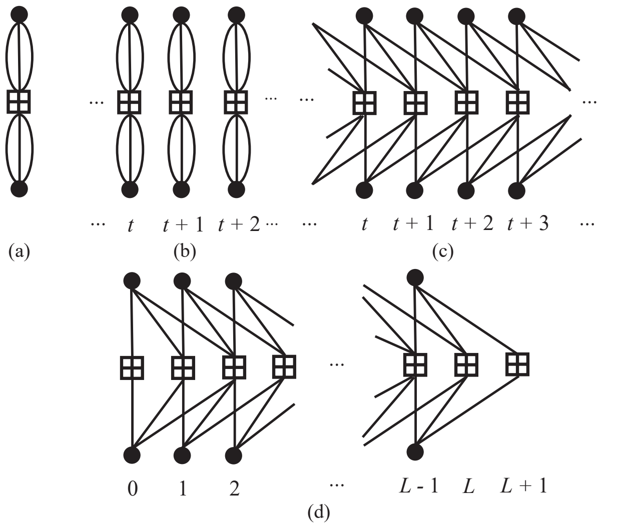

Fig. 1. (a) A block protograph for $\mathbf{B} = [3, 3]$, (b) an infinite sequence of uncoupled protographs, (c) coupling all the protographs over time, and (d) a terminated chain of L coupled protographs.

$\mathbf{B}_0, \mathbf{B}_1, \ldots, \mathbf{B}_\omega$, such that

$$\mathbf{B}(r, s) = \sum_{i=0}^{\omega} \mathbf{B}_i(r, s), \tag{1}$$

where ω is the coupling width of the constructed SC-LDPC code. Fig. 1(c) shows an example of $\omega = 2$ and $\mathbf{B}_0 = \mathbf{B}_1 = \mathbf{B}_2 = [1, 1]$. Consequently, an infinite chain of coupled protographs that is formed, i.e., $\mathbf{B}_{[-\infty,+\infty]}$ extends infinitely over time. Note that a practical SC-LDPC code has finite length, which can be constructed from the coupled protograph of finite length as illustrated by Fig. 1(d). The terminated coupled protograph is obtained by coupling L block protographs, where L is called the coupling length. This coupled protograph can be interpreted by the following base matrix

$$\mathbf{B}_{[0,L-1]} = \begin{bmatrix} \mathbf{B}_0 & & & \\ \mathbf{B}_1 & \mathbf{B}_0 & & \\ \vdots & \mathbf{B}_1 & \ddots & \\ \mathbf{B}_\omega & \vdots & & \mathbf{B}_0 \\ & \mathbf{B}_\omega & & \mathbf{B}_1 \\ & & \ddots & \vdots \\ & & & \mathbf{B}_\omega \end{bmatrix}, \tag{2}$$

which has size $(L+\omega)n_c \times Ln_v$. The designed rate of the code is $R_{\mathrm{SC}}^{(L)} = 1 - \frac{(L+\omega)n_c}{Ln_v}$, where $\lim_{L\to+\infty} R_{\mathrm{SC}}^{(L)} = 1 - \frac{n_c}{n_v}$.

Let $\mathbf{P}_a$ denote an $a \times a$ permutation matrix and $\mathbf{I}_a$ an $a \times a$ identity matrix. The Tanner graph of a finite SC-LDPC code can be obtained by applying the M-fold graph lifting to the above coupled protograph. Similarly, parity-check matrix $\mathbf{H}_{[0,L-1]}$ of the code can be obtained by the M-fold matrix expansion over $\mathbf{B}_{[0,L-1]}$. Each non-zero entry in $\mathbf{B}_{[0,L-1]}$ is replaced by a sum of $\mathbf{B}(r, s)$ permutation matrices $\mathbf{P}_M$ and the zero entries are replaced by the $M \times M$ all zero matrices. Let $N_c = n_c M$ and $N_v = n_v M$, $\mathbf{H}_{[0,L-1]}$ is of size $(L + \omega)N_c \times LN_v$. Let $\mathbf{H}_i(t)$ denote the expanded outcome of $\mathbf{B}_i$ at time instant t, where $0 \leq t \leq L + \omega - 1$. After the M-fold graph lifting, the parity-check matrix $\mathbf{H}_{[0,L-1]}$ is

$$\mathbf{H}_{[0,L-1]} = \begin{bmatrix} \mathbf{H}_0(0) & & & \\ \mathbf{H}_1(1) & \mathbf{H}_0(1) & & \\ \vdots & \mathbf{H}_1(2) & \ddots & \\ \mathbf{H}_\omega(\omega) & \vdots & & \mathbf{H}_0(L-1) \\ & \mathbf{H}_\omega(\omega+1) & & \mathbf{H}_1(L) \\ & & \ddots & \vdots \\ & & & \mathbf{H}_\omega(L+\omega-1) \end{bmatrix}, \tag{3}$$

where $\mathbf{H}_i(t)$ can be expressed as

$$\mathbf{H}_i(t) = \begin{bmatrix} h_i^{(0,0)}(t) & \cdots & h_i^{(0,N_v-1)}(t) \\ \vdots & & \vdots \\ h_i^{(N_c-1,0)}(t) & \cdots & h_i^{(N_c-1,N_v-1)}(t) \end{bmatrix} \tag{4}$$

and $h_i^{(\lambda,\mu)}(t) \in \{0, 1\}$ is the row-λ, column-μ entry. Note that it is assumed $\mathbf{H}_0(t)$ is full rank.

B. Locally Systematic Encoding of SC-LDPC Codes

Since the SC-LDPC code is decoded by the SWD, realizing its locally systematic encoding will be necessary so that SWD outcome can be further decoded by the outer code. Let

$$\underline{U}_{[0,L-1]} = (\underline{U}_0, \underline{U}_1, \ldots, \underline{U}_{L-1}) \tag{5}$$

denote a sequence of L outer codewords in binary form, where $\underline{U}_t$ = $(U_{t,0}, U_{t,1}, \ldots, U_{t,N_v-N_c-1})$ and $0 \leq t \leq L-1$. Further let

$$\underline{V}_{[0,L-1]} = (\underline{V}_0, \underline{V}_1, \ldots, \underline{V}_{L-1}) \tag{6}$$

denote the corresponding terminated SC-LDPC codeword, where $\underline{V}_t = (V_{t,0}, V_{t,1} \ldots, V_{t,N_v-1})$. Therefore,

$$\underline{V}_{[0,L-1]} \cdot \mathbf{H}_{[0,L-1]}^T = \underline{0}, \tag{7}$$

where $\mathbf{H}_{[0,L-1]}^T$ can be defined as in (8). In particular, $\underline{V}_{[0,L-1]}$ should satisfy

$$\underline{V}_t \mathbf{H}_0^T(t) + \underline{V}_{t-1} \mathbf{H}_1^T(t) + \cdots + \underline{V}_{t-\omega} \mathbf{H}_\omega^T(t) = \underline{0} \tag{9}$$

for $t = 0, 1, \ldots, L - 1$. However, for $t \geq L$, termination bits $V_{LN_v}, V_{LN_v+1}, \ldots, V_{(L+\omega+1)N_v-1}$ are needed. They can be generated based on the extended $(\omega + 1)N_v$ rows from $\mathbf{H}_{[0,L-1]}^T$. Details of their generation can be referred to [11].

Since $\mathbf{H}_0^T(t)$ is full rank, (9) can be utilized to generate $\underline{V}_t$. In order to realize locally systematic encoding, we let the last N_c rows of $\mathbf{H}_0^T(t)$ form an identity submatrix. As a result,

$$V_{t,j} = U_{t,j}, \tag{10}$$

for $j = 0, 1, \ldots, N_v - N_c - 1$. Otherwise, for $j = N_v -$

$$\mathbf{H}_{[0,L-1]}^T = \begin{bmatrix} \mathbf{H}_0^T(0) & \mathbf{H}_1^T(1) & \cdots & \mathbf{H}_\omega^T(\omega) & & & & \\ & \mathbf{H}_0^T(1) & \cdots & \mathbf{H}_{\omega-1}^T(\omega) & \mathbf{H}_\omega^T(\omega+1) & & & \\ & & \ddots & & & \ddots & & \\ & & & \mathbf{H}_0^T(L-1) & \mathbf{H}_1^T(L) & \cdots & \mathbf{H}_\omega^T(L+\omega-1) \end{bmatrix}. \quad (8)$$

$N_c, N_v - N_c + 1, \ldots, N_v - 1,$

$$V_{t,j} = \sum_{\mu=0}^{N_v-N_c-1} V_{t,\mu} \cdot h_0^{(\lambda,\mu)}(t) + \sum_{i=1}^{\omega} \sum_{\mu=0}^{N_v-1} V_{t-i,\mu} \cdot h_i^{(\lambda,\mu)}(t) \quad (11)$$

where $\lambda = j - (N_v - N_c)$.

III. RS-SC-LDPC Codes

This section introduces the proposed RS-SC-LDPC codes. We will also introduce the design rules of the inner code so that it has a large girth, and realizes locally systematic encoding.

A. The Concatenated Coding Scheme

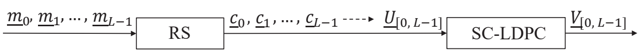

Fig. 2. Block diagram of the RS-SC-LDPC code.

Block diagram of RS-SC-LDPC codes is shown as in Fig. 2. Let $\mathbb{F}_q$ denote the finite field of size q. For practical concern, we assume that the RS codes are defined in a binary extension field, i.e., $q = 2^p$, where p is a positive integer. An (n, k) RS code defined over $\mathbb{F}_{2^p}$ has length $n = 2^p - 1$ and dimension k, where $k < n$. In this work, the RS code is decoded by the Berlekamp-Massey (BM) algorithm [12], which can correct at most $\lfloor \frac{n-k}{2} \rfloor$ symbol errors.

For the finite length concatenated code, the inner code has a coupling length of L. Given L message vectors $\underline{m}_t = (m_{t,0}, m_{t,1}, \ldots, m_{t,k-1}) \in \mathbb{F}_{2^p}^k$, where $t = 0, 1, \ldots, L-1$, each message vector can be encoded by an (n, k) RS code. Message $\underline{m}_t$ can be written as a message polynomial

$$m_t(x) = m_{t,0} + m_{t,1}x + \cdots + m_{t,k-1}x^{k-1}. \quad (12)$$

The generator polynomial of an (n, k) RS code is

$$g(x) = (x - \sigma^1)(x - \sigma^2) \cdots (x - \sigma^{n-k}), \quad (13)$$

where σ is a primitive element of $\mathbb{F}_{2^p}$. The codeword $\underline{c}_t = (c_{t,0}, c_{t,1}, \ldots, c_{t,n-1}) \in \mathbb{F}_{2^p}^n$ can be generated by

$$\begin{aligned} c_t(x) &= x^{n-k}m_t(x) + ((x^{n-k}m_t(x)) \bmod g(x)) \\ &= c_{t,0} + c_{t,1}x + \cdots + c_{t,n-1}x^{n-1}. \end{aligned} \quad (14)$$

Then L RS codewords $\underline{c}_0, \underline{c}_1, \cdots, \underline{c}_{L-1}$ are converted into a binary RS codeword sequence, which is the input of the SC-LDPC encoder. It can be written as

$$\underline{U}_{[0,L-1]} = (\underline{U}_0, \underline{U}_1, \ldots, \underline{U}_{L-1}),$$

where $\underline{U}_t \in \mathbb{F}_2^{np}$ is the binary representation of $\underline{c}_t$ and $\underline{U}_t = (U_{t,0}, U_{t,1}, \ldots, U_{t,N_v-N_c-1})$. Finally, the finite length concatenated codeword is generated by (10) and (11), which can be written as

$$\underline{V}_{[0,L-1]} = (\underline{V}_0, \underline{V}_1, \ldots, \underline{V}_{L-1}),$$

where $\underline{V}_t = (V_{t,0}, V_{t,1}, \ldots, V_{t,N_v-1})$. The encoder is locally systematic and therefore $\underline{V}_t = [\underline{V}_t^{(0)}, \underline{V}_t^{(1)}]$, where $\underline{V}_t^{(0)} = \underline{U}_t$, and $\underline{V}_t^{(1)}$ is the parity-check portion of length N_c.

B. Design of Inner Code

It is known that the girth of the Tanner graph of $\mathbf{H}_{[0,L-1]}$ is lower bounded by that of the protograph of $\mathbf{B}_{[0,L-1]}$. Hence, we can first eliminate 4-cycles in $\mathbf{B}_{[0,L-1]}$ and then construct $\mathbf{H}_{[0,L-1]}$ with a larger girth by applying a systematic M-fold graph lifting using the Fossorier condition [13]. In $\mathbf{B}_{[0,L-1]}$, 4-cycles may exist in several patterns as illustrated by Fig. 3. It can be seen that 4-cycles can be contained in a submatrix of

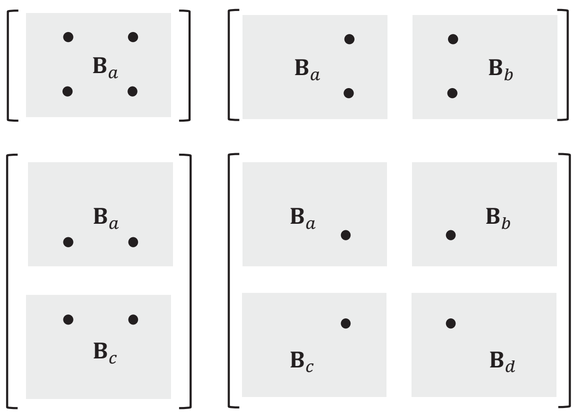

Fig. 3. All 4-cycle patterns in $\mathbf{B}_{[0,L-1]}$, where $\mathbf{B}_a$, $\mathbf{B}_b$, $\mathbf{B}_c$ and $\mathbf{B}_d$, $a, b, c, d \in \{0, 1, \ldots, \omega\}$, are the submatrices in $\mathbf{B}_{[0,L-1]}$. The non-zero entries are indicated by solid circles.

$\mathbf{B}_{[0,L-1]}$, or two submatrices of the same row (resp. column) of $\mathbf{B}_{[0,L-1]}$, or four submatrices that appear in a rectangular array of $\mathbf{B}_{[0,L-1]}$. In order to design $\mathbf{B}_{[0,L-1]}$ without 4-cycles, i.e., the girth is at least six, we need to define the following representative block $\mathbf{B}_{\mathrm{R}}$ as follows

$$\mathbf{B}_{\mathrm{R}} \triangleq \begin{bmatrix} \mathbf{B}_\omega & \mathbf{B}_{\omega-1} & \cdots & \mathbf{B}_0 \\ & \mathbf{B}_\omega & \cdots & \mathbf{B}_1 \\ & & \ddots & \vdots \\ & & & \mathbf{B}_\omega \end{bmatrix}. \quad (15)$$

It is of size $(\omega + 1)n_c \times (\omega + 1)n_v$. It contains all possible submatrices patterns of Fig. 3, which can lead to 4-cycles in $\mathbf{B}_{[0,L-1]}$. If $\mathbf{B}_{\mathrm{R}}$ does not contain 4-cycles, neither will $\mathbf{B}_{[0,L-1]}$. In order to design $\mathbf{B}_{\mathrm{R}}$, it can be further decomposed into the constituent block $\mathbf{B}_\mathrm{C}$ and the excluded patterns $\mathbf{B}_\mathrm{E}^{(l)}$, where $l \in \{1, 2, \ldots, n_\mathrm{E}\}$ and n_E is the total number of the excluded patterns. $\mathbf{B}_\mathrm{C}$ is obtained by forming a rectangular matrix from $\mathbf{B}_\mathrm{R}$ with at least two submatrices in each row and each column, containing $\mathbf{B}_0$ in its upper-right corner and one of the $\mathbf{B}_\omega$s along the diagonal in its lower-left corner. It is defined as

$$\mathbf{B}_\mathrm{C} \triangleq \begin{bmatrix} \mathbf{B}_{\beta-1} & \cdots & \mathbf{B}_1 & \mathbf{B}_0 \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{B}_\omega & \cdots & \mathbf{B}_\alpha & \mathbf{B}_{\alpha-1} \end{bmatrix}, \tag{16}$$

where $\omega = \alpha + \beta - 2$ and $\alpha, \beta > 1$, with size $\alpha n_c \times \beta n_v$. The excluded patterns $\mathbf{B}_\mathrm{E}^{(l)}$ are defined as

$$\mathbf{B}_\mathrm{E}^{(1)} = \begin{bmatrix} \mathbf{B}_\omega & \mathbf{B}_0 \end{bmatrix}, \ \mathbf{B}_\mathrm{E}^{(2)} = \begin{bmatrix} \mathbf{B}_0 \\ \mathbf{B}_\omega \end{bmatrix},$$

$$\mathbf{B}_\mathrm{E}^{(l)} = \begin{bmatrix} \mathbf{B}_{a_l} & \mathbf{B}_{b_l} \\ \mathbf{B}_{c_l} & \mathbf{B}_{d_l} \end{bmatrix}, \ l = 3, 4, \ldots, n_\mathrm{E}, \tag{17}$$

where $a_l, b_l, c_l, d_l \in \{0, 1, \ldots, \omega\}$ and n_E depends on ω and the chosen $\mathbf{B}_\mathrm{C}$. Note that all excluded patterns appear in $\mathbf{B}_\mathrm{R}$ but do not appear in $\mathbf{B}_\mathrm{C}$.

With the above definition, a two stage design approach for $\mathbf{B}_{[0,L-1]}$ has been proposed in [14]. In Design Stage I, the submatrices are initialized based on the excluded patterns $\mathbf{B}_\mathrm{E}^{(l)}$, ensuring that there is no 4-cycle in $\mathbf{B}_\mathrm{E}^{(l)}$. In Design Stage II, the initialized submatrices are modified to remove the remaining 4-cycles in $\mathbf{B}_\mathrm{C}$. Ensuring $\mathbf{B}_\mathrm{C}$ and all $\mathbf{B}_\mathrm{E}^{(l)}$ have no 4-cycles, the designed $\mathbf{B}_{[0,L-1]}$ has a girth of at least six. $\mathbf{H}_{[0,L-1]}$ of girth of at least eight is further generated using the M-fold graph lifting.

In this concatenated coding scheme, encoding of the inner SC-LDPC code is locally systematic. Hence, in order to obtain $\underline{V}_t$, we only need to focus on the submatrix in which the information vector $\underline{U}_t$ participates at the time instant t, i.e., $\mathbf{H}_0(t)$. In $\mathbf{H}_0(t)$, each information bit and check bit should participate in at least one parity-check equation so that they can be recovered by SWD. At the beginning of SWD, $\mathbf{H}_0(t)$ should has a row weight of at least two. Each parity-check equation of $\mathbf{H}_0(t)$ should contain at least one information bit and one check bit. Hence, Design Stage I of [14] should be modified. Firstly, let $\mathbf{\Phi}_{a\times b}$ denote an $a \times b$ binary matrix with a minimum column weight of one and with a maximum row weight as small as possible. Secondly, let $\mathbf{\Xi}_{a\times b}$ denote an $a\times b$ binary matrix with a minimum row weight of one and with a maximum column weight as small as possible. The modified Design Stage I can be presented as the following steps.

Step 1 ensures that $\mathbf{B}_0$ has the minimum row weight of two, i.e., the minimum row weight of $\mathbf{B}_0^{(0)}$ and $\mathbf{B}_0^{(1)}$ is one. Moreover, the locally systematic encoding property can be realized. The minimum column weight of $\mathbf{B}_0^{(0)}$ and $\mathbf{B}_0^{(1)}$ is designed as one so that each information bit or parity-check bit participates in at least one parity-check equation. So that, the coded bits can be recovered uniquely. If $n_v - n_c \geq n_c$, the minimum column weight of $\mathbf{B}_0^{(0)}$ is one and if $n_v - n_c < n_c$, the minimum row weight of $\mathbf{B}_0^{(0)}$ is one. Further, $\mathbf{B}_0^{(1)}$ is designed as an identity matrix in order to make sure that $\mathbf{B}_0$ is full rank. Consequently, its expanded outcome $\mathbf{H}_0(t)$ is also full rank. Note that the design of $\mathbf{B}_\omega$ in **Step 2** does not have to ensure the minimum row weight of two. This is because the termination bits are redundant, and do not need to be decoded.

Design Stage 1 Initialize the Submatrices

Step 1: Let

$$\mathbf{B}_0 = [\mathbf{B}_0^{(0)} \ \mathbf{B}_0^{(1)}], \tag{18}$$

where $\mathbf{B}_0^{(1)}$ is an identity matrix of size $n_c \times n_c$. If $n_v - n_c \geq n_c$, $\mathbf{B}_0^{(0)}$ is defined as

$$\mathbf{B}_0^{(0)} = \begin{bmatrix} \mathbf{P}_{n_c} & \mathbf{\Phi}_{n_c \times (n_v - 2n_c)} \end{bmatrix}, \tag{19}$$

where $\mathbf{\Phi}_{n_c\times(n_v-2n_c)}$ is chosen such that there is no 4-cycle in $\mathbf{B}_0^{(0)}$; otherwise, $\mathbf{B}_0^{(0)}$ is defined as

$$\mathbf{B}_0^{(0)} = \begin{bmatrix} \mathbf{P}_{n_v - n_c} \\ \mathbf{\Xi}_{(2n_c - n_v)\times(n_v - n_c)} \end{bmatrix}, \tag{20}$$

where $\mathbf{\Xi}_{(2n_c-n_v)\times(n_v-n_c)}$ is chosen such that there is no 4-cycle in $\mathbf{B}_0^{(0)}$.

Step 2: Initialize $\mathbf{B}_\omega$ such that neither $\mathbf{B}_\omega$, nor $\mathbf{B}_\mathrm{E}^{(1)}$ and $\mathbf{B}_\mathrm{E}^{(2)}$ contain 4-cycles.

Step 3: Initialize other submatrices, i.e., $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_{\omega-1}$, such that $\mathbf{B}(r,s) = \sum_{i=0}^{\omega} \mathbf{B}_i(r,s)$ and there is no 4-cycle in any of the submatrices or excluded patterns $\mathbf{B}_\mathrm{E}^{(l)}$ ($l = 3, 4, \ldots, n_\mathrm{E}$).

Finally, let $\mathbf{I}_a^{(\theta)}$ denote the shifted identity matrix with each row of $\mathbf{I}_a$ cyclically shifted to the left by θ positions. We apply the M-fold graph lifting on the designed $\mathbf{B}_{[0,L-1]}$. Using the Fossorier condition [13], 6-cycles can be removed by lifting non-zero entries with different $\mathbf{I}_M^{(\theta)}$. Hence, we can construct the inner code with girth $g \geq 8$.

C. Locally Systematic Encoding of Inner Code

Since $\mathbf{H}_0(t)$ is generated by applying the M-fold graph lifting on $\mathbf{B}_0$ at time instant t, the last N_c rows of $\mathbf{H}_0(t)$ does not necessarily form an identity matrix after the lifting. In order to define the locally systematic encoding of the designed inner code, the following proposition is needed.

Proposition 1: $\mathbf{I}_M^{(M-\theta)}$ is the transpose of $\mathbf{I}_M^{(\theta)}$.

The above proposition helps determine the parity-check equation which generates the corresponding check bit in the encoding. Let ρ denote the column index of $\mathbf{H}_0^\mathrm{T}(t)$, where $\rho = 0, 1, \ldots, N_c - 1$, corresponding to the parity-check equation which generates the check bit $V_{t,j}$ and $j = N_v - N_c, N_v - N_c + 1, \ldots, N_v - 1$. Denote a permutation matrix $\mathcal{P}$ of size $M \times M$ as the targeted partitioned matrix with the non-zero element corresponding to check bit $V_{t,j}$'s location in matrix $\mathbf{H}_0^\mathrm{T}(t)$. Let ρ_c, where $\rho_c = 0, 1, \ldots, N_c - 1$, denote the location index of the check bit $V_{t,j}$ in $\underline{V}_t^{(1)}$, which can be given by

$$\rho_c = j - (N_v - N_c). \tag{21}$$

Further let

$$\rho_0 = \lfloor \frac{\rho_c}{M} \rfloor M. \tag{22}$$

Given a shifting factor θ of the targeted partitioned matrix, $\mathcal{P} = (\mathbf{I}_M^{(\theta)})^T$. Based on Proposition 1,

$$\rho = \rho_0 + (\rho_c + \theta) \bmod M. \tag{23}$$

It means that the value of ρ is the sum of the column index of the first column in the targeted partitioned matrix $\mathcal{P}$ in $\mathbf{H}_0^T(t)$ and the position of the non-zero entry corresponding to the check bit $V_{t,j}$ relative to the first column of $\mathcal{P}$. Note that when the shifting factor $\theta = 0$, the column index ρ corresponding to check bit $V_{t,j}$ is the location index of check bit $V_{t,j}$ in $\underline{V}_t^{(1)}$, i.e., $\rho = \rho_c$. Therefore, the locally systematic encoding follows:

$$V_{t,j} = U_{t,j}, \tag{24}$$

if $j = 0, 1, \ldots, N_v - N_c - 1$, and

$$V_{t,j} = \sum_{\mu=0}^{N_v - N_c - 1} V_{t,\mu} \cdot h_0^{(\rho,\mu)}(t) + \sum_{i=1}^{\omega} \sum_{\mu=0}^{N_v - 1} V_{t-i,\mu} \cdot h_i^{(\rho,\mu)}(t), \tag{25}$$

if $j = N_v - N_c, N_v - N_c + 1, \ldots, N_v - 1$.

***Example** 1:* The following example illustrates the above systematic encoding process. Given $\mathbf{H}_{[0,L-1]}^T$ as shown by Fig. 4, we demonstrate the calculation of $V_{1,14}$.

First, we find that the location index of $V_{1,14}$ in $\underline{V}_1^{(1)}$ is 5, i.e., $\rho_c = 5$. Based on (22), we can obtain the column index of the first column in targeted partitioned matrix $\mathcal{P}$ in $\mathbf{H}_0^T(1)$, which is shown as the dashed box, i.e., $\rho_0 = 3$. Assume that $\theta = 1$, the position of the entry 1 corresponding to $V_{1,14}$ relative to the first column in the dashed box is 0 and then we can calculate that $\rho = 3$. Finally, the parity-check equation of column 3 at time instant $t = 1$ is $V_{1,14} = V_{0,12} + V_{1,0} = 0$.

IV. Performance Analysis

As shown in Fig. 5, for the concatenated code, the SWD and the BM algorithms are applied to decode the inner and outer codes, respectively. Let W denote the size of decoding window which contains W block protographs, where $\omega + 1 \leqslant W \leqslant L$. The SWD performs log-BP decoding with a window size of W at each decoding instant, aiming at recover only the targeted symbols. For the finite length code, the entire codeword will be recovered over L decoding instants.

For each decoding instant, log-likelihood ratios (LLRs) of the targeted symbols within each window will be estimated. Hard decisions are made based on the estimated LLRs and further decoded by the outer RS code. If the RS code can be decoded, the targeted symbols are known and their LLR values become deterministic, i.e., $+\infty$ or $-\infty$. They participate into the SWD therein, improving the SWD performance. Otherwise, the LLR values are not adjusted.

We compare the performance between SC-LDPC codes and RS-SC-LDPC codes. The designed inner codes have girth

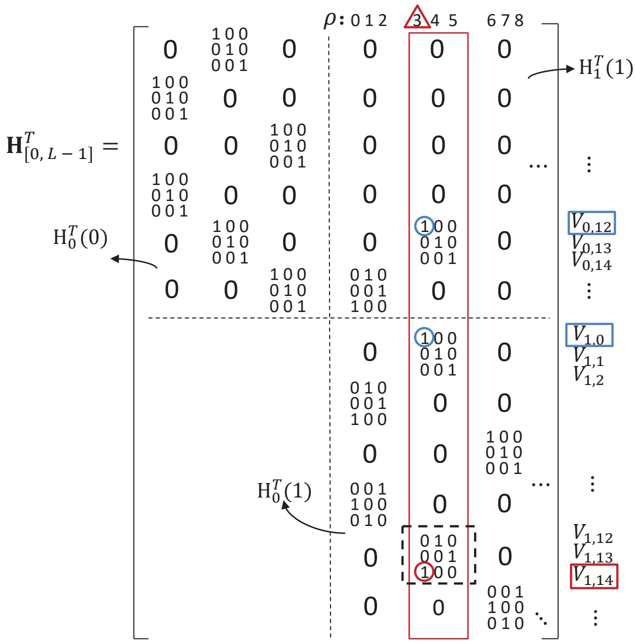

Fig. 4. Part of $\mathbf{H}_{[0,L-1]}^T$ with $M = 3$, $\omega = 3$, $n_c = 3$ and $n_v = 6$.

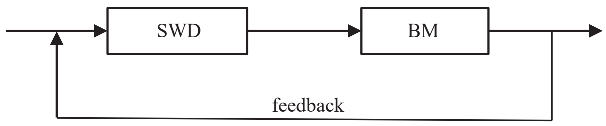

Fig. 5. Block diagram of the SWD-BM decoder.

eight with $(n_c, n_v) = (3, 6)$, $M = 126$, $\omega = 3$ and $L = 60$. Simulations were performed over the AWGN channel using BPSK modulation.

TABLE I
The Ratio of Blocks with Symbol Errors more than $\lfloor \frac{n-k}{2} \rfloor$

RS Code	**SNR**		
	1.6	**1.7**	**1.8**
(63, 61)	0.00375%	0.00069%	0.00044%
(63, 59)	0.14652%	0.00667%	0.00058%

We first consider the choice of the outer RS code. We examine the error patterns of the concatenated code after SWD in different SNRs (up to 1.8 dB) in order to see if the RS code is adequate to combat the errors. Note that the RS code can correct at most $\lfloor \frac{n-k}{2} \rfloor$ symbol errors by using the BM algorithm. As shown in **Table I**, when $W = 20$, the ratio of inner blocks whose symbol errors are more than 1, tends to be stable in the high SNR region when concatenated with the (63, 61) RS code. It indicates that the error floor of the concatenated code still exists when the outer code is the (63, 61) RS code. A lower-rate RS code will be chosen as the candidate for the outer code. It can be seen that there is no error floor when SNR $\leq$ 1.8 dB by concatenating a (63, 59)

RS code. Thus the (63, 59) RS code is chosen as the outer code for the following concatenated codes.

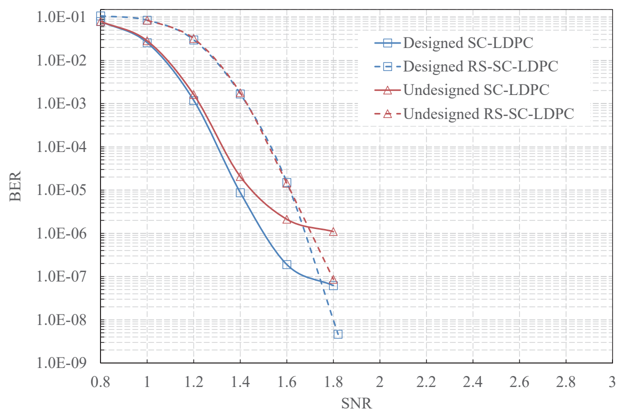

Fig. 6. Performance of designed and undesigned SC-LDPC codes and RS-SC-LDPC codes.

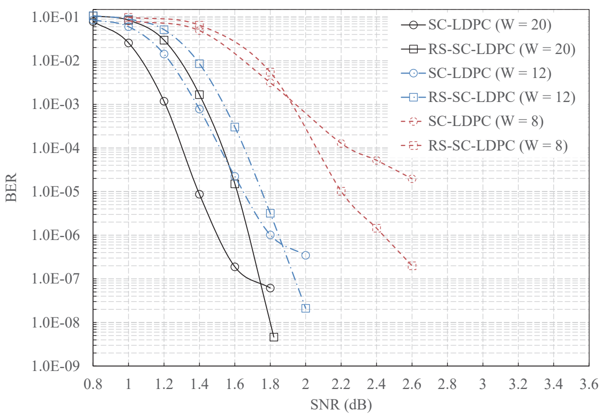

Fig. 7. Performance of designed SC-LDPC codes with no outer code and with a (63, 59) RS outer code.

Fig. 6 shows the performance of designed and undesigned SC-LDPC codes and their corresponding RS-SC-LDPC codes. The undesigned inner codes were constructed from the base matrix $\mathbf{B} = [3\ 3]$ with $\mathbf{B}_0 = [1\ 1]$, $\mathbf{B}_1 = [1\ 0]$, $\mathbf{B}_2 = [0\ 1]$ and $\mathbf{B}_3 = [1\ 1]$, and $M = 378$. Designed codes follow the design rules in Section III with $W = 20$ in all cases and the code rates of the SC-LDPC code and concatenated code are 0.47 and 0.44, respectively. We see that the concatenated code helps recover the error floor from the original SC-LDPC code. The designed codes also outperform the undesigned ones thanks to their larger girth property. However, concatenating with outer codes leads to rate loss. This yields a slight degradation in waterfall performance. Fig. 7 shows the performance of SC-LDPC codes and corresponding RS-SC-LDPC codes with different window sizes W. It shows that when the window size is too small, e.g., $W = 8$, concatenating an RS code cannot completely remove the error floor. This is due to the fact that the chosen (63, 59) RS code cannot eliminate all errors yielded by the inner SWD.

V. Conclusions

This paper has proposed the RS-SC-LDPC concatenated codes, realizing both low message recovery latency and high decoding performance for further data streaming. We present the design rules and the locally systematic encoding of the inner codes. With the proposed SWD-BM decoding, our simulation results show that the proposed codes outperform the SC-LDPC codes, removing the error floor when the window size is sufficiently large.

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