Chapter 2 Channel Capacity

- 2.1 Introduction
- 2.2 Binary Symmetric Channel
- 2.3 Binary Erasure Channel
- 2.4 Additive White Gaussian Noise Channel
- 2.5 Fading Channels

§ 2.1 Introduction

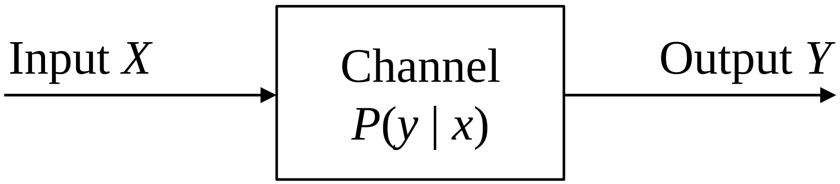
- In a communication system, with the observation of Y, we aim to recover X.
- Mutual Information $I(X,Y) = H(X) - H(X|Y)$

$$= H(Y) - H(Y|X)$$

  It defines the amount of uncertainty about X that has been reduced thanks to the knowledge of Y, and vise versa. This uncertainty discrepancy is introduced by the channel.

- Channel capacity describes the channel's best capability in reducing the uncertainty.

§ 2.1 Introduction



- Let the realization of input X and output Y be x and y, respectively.
- Channel transition probability $P(y \mid x)$: knowing x was transmitted, the probability of observing y. It defines the quality of channel.
- Channel Capacity

$$C = \max_{P(x)}\{I(X,Y)\}$$

The maximum mutual information $I(X,Y)$ that can be realized over all distribution of the input $P(x)$.

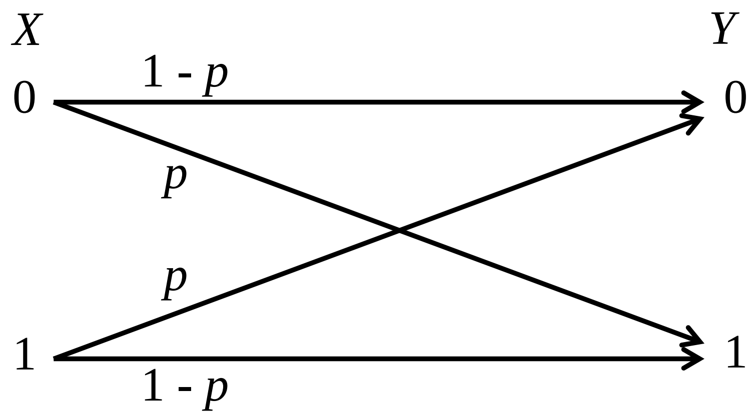
§ 2.1 Introduction

- Channel Capacity: $C = \max\limits_{P(x)}\{I(X,Y)\}$
- In a wireless communication system, it is the maximum number of information bits that can be carried by a modulated symbol such that the information can be recovered with an arbitrarily low probability of error.
- To realize this reliable communications, channel coding is needed. Given k information symbols (or bits), redundancy is added to obtain n ($n > k$) codeword symbols (or bits). The code rate is $r = \frac{k}{n}$. Using binary modulation, e.g., BPSK, reliable communications is possible if $r < C$.

§ 2.1 Introduction

- Why input distribution $P(x)$ matters?
- Consider the data transmission as human flows from Shenzhen to Hong Kong

§ 2.2 Binary Symmetric Channel (BSC)

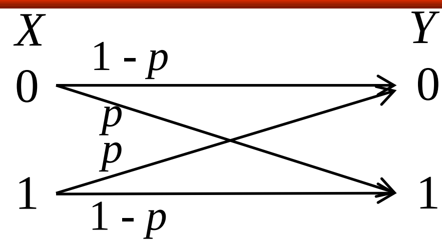


- Input: 0 1 0 0 0 1 1 0 1 0 ...

  Output: 0 1 1 1 0 0 1 0 0 0 ...
- Input and output are discrete
- $P(y = 1|x = 0) = P(y = 0|x = 1) = p$

  $P(y = 0|x = 0) = P(y = 1|x = 1) = 1 - p$
- It is the simplest model of channel that introduces errors. Many wireless channels can be abstracted to BSC.

§ 2.2 Binary Symmetric Channel (BSC)



- Analytic intuition

$$I(X,Y) = H(Y) - H(Y|X)$$

$I(X,Y)$ will be maximized if $H(Y)$ is maximized and $H(Y|X)$ is minimized.

(1) $H(Y) \leq 1$.

(2) $H(Y|X) = -\sum_{x\in X}\sum_{y\in Y} P(x,y)\log_2 P(y|x)$

$$= -\sum_{x\in X}\sum_{y\in Y} P(y|x)P(x)\log_2 P(y|x)$$

$$= -P(x=0)\sum_{y\in\{0,1\}} P(y|x=0)\log_2 P(y|x=0)$$

$$-P(x=1)\sum_{y\in\{0,1\}} P(y|x=1)\log_2 P(y|x=1)$$

$$= -P(x=0)((1-p)\log_2(1-p) + p\log_2 p) - P(x=1)(p\log_2 p + (1-p)\log_2(1-p))$$

$$= -(1-p)\log_2(1-p) - p\log_2 p$$

- When $P(x=0) = P(x=1) = \frac{1}{2}$, $H(Y) = 1$ and

$$C = 1 - H(Y|X) \text{ bits/symbol.}$$

§ 2.2 Binary Symmetric Channel (BSC)

- Intuition: If 0 and 1 experience the same degree of channel impairment, i.e., $P(y=1|x=0)=P(y=0|x=1)$, there is no need to prioritize either 0 or 1 for transmission and $P(x=0)=P(x=1)=\frac{1}{2}$.
- $C=1-H(Y|X)$, if $P(x=0)=P(x=1)=\frac{1}{2}$.
- $H(Y|X)=-P(y=0|x=0)\cdot\frac{1}{2}\cdot\log_2 P(y=0|x=0)$

$$=-P(y=1|x=0)\cdot\frac{1}{2}\cdot\log_2 P(y=1|x=0)$$

$$=-P(y=0|x=1)\cdot\frac{1}{2}\cdot\log_2 P(y=0|x=1)$$

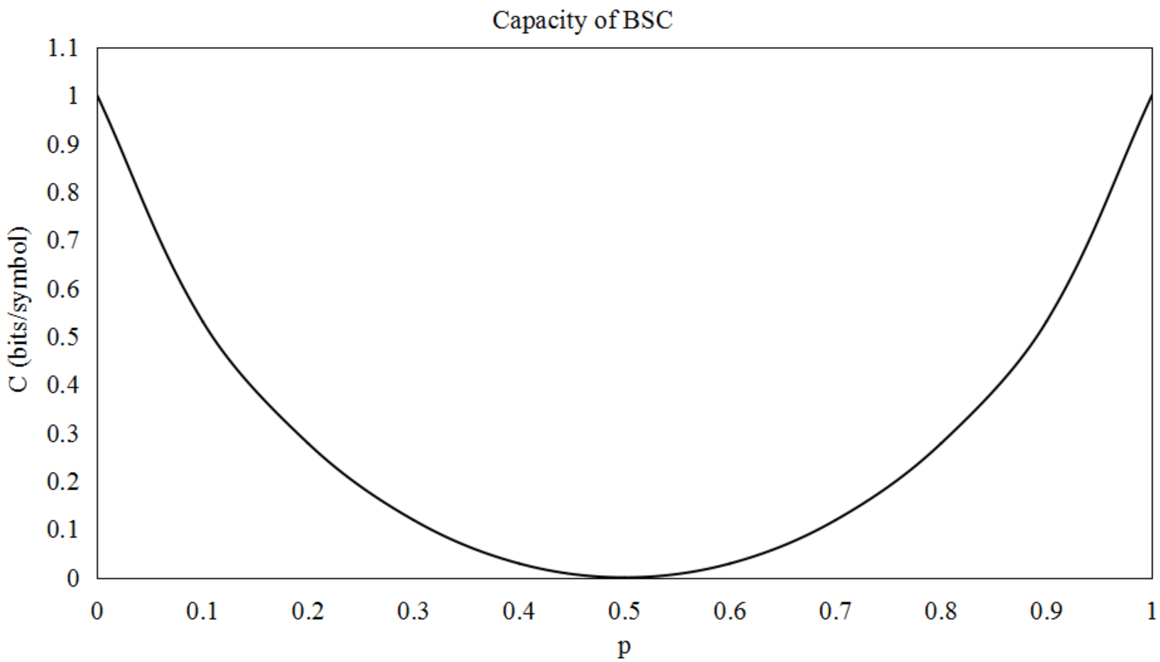
$$=-P(y=1|x=1)\cdot\frac{1}{2}\cdot\log_2 P(y=1|x=1)$$

$$=-p\log_2 p-(1-p)\log_2(1-p)$$

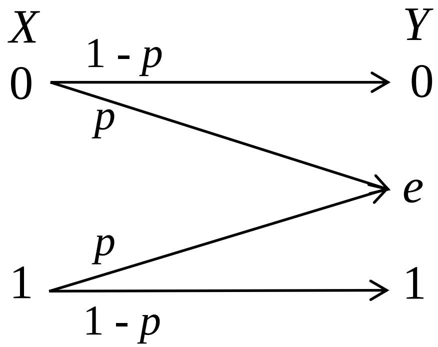
- $C=1+p\log_2 p+(1-p)\log_2(1-p)$ bits/symbol

§ 2.2 Binary Symmetric Channel (BSC)

- $C = 1 + p\log_2 p + (1-p)\log_2(1-p)$ bits/symbol

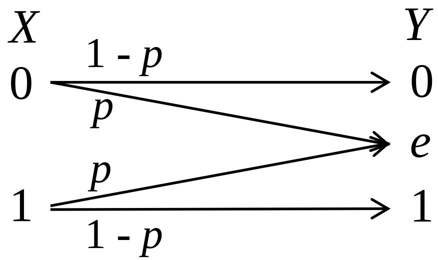


§ 2.3 Binary Erasure Channel (BEC)



- Input: 1 0 0 1 0 1 0 1 0 0 ...
  Output: 1 e 0 1 0 e 0 e 0 e ...
- $P(y = e|x = 0) = P(y = e|x = 1) = p$
  $P(y = 0|x = 0) = P(y = 1|x = 1) = 1 - p$
- It is a channel model often used in computer networks. Data packets are either perfectly received or lost.

§ 2.3 Binary Erasure Channel (BEC)



- Similar to the analytic intuition of BSC, channel capacity is reached when $P(x=0) = P(x=1) = \frac{1}{2}$.
- Hence $C = H(Y) - H(Y|X)$ bits/symbol
- Since $P(y=0) = P(y=1) = \frac{1}{2}(1-p)$ and $P(y=e) = p$

$$H(Y) = -\sum_{y \in Y} P(y)\log_2 P(y)$$

$$= -P(y=0)\log_2 P(y=0) - P(y=e)\log_2 P(y=e) - P(y=1)\log_2 P(y=1)$$

$$= -\frac{1}{2}(1-p)\log_2\frac{1}{2}(1-p) - p\log_2 p - \frac{1}{2}(1-p)\log_2\frac{1}{2}(1-p)$$

$$= -(1-p)\log_2\frac{1}{2}(1-p) - p\log_2 p$$

§ 2.3 Binary Erasure Channel (BEC)

- $H(Y|X) = -\sum_{x\in X}\sum_{y\in Y} P(y|x)P(x)\log_2 P(y|x)$

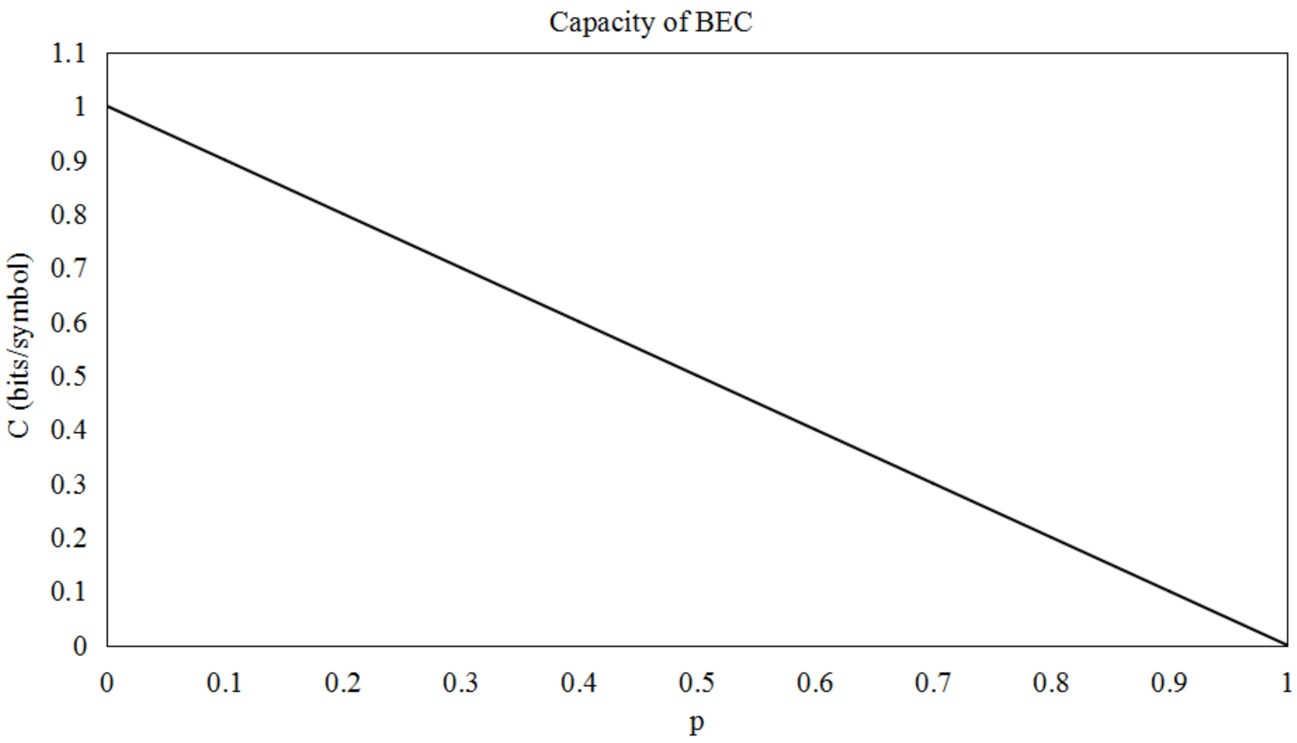
$$= -P(y=0|x=0)\cdot\frac{1}{2}\cdot\log_2 P(y=0|x=0)$$
$$-P(y=1|x=1)\cdot\frac{1}{2}\cdot\log_2 P(y=1|x=1)$$
$$-P(y=e|x=0)\cdot\frac{1}{2}\cdot\log_2 P(y=e|x=0)$$
$$-P(y=e|x=1)\cdot\frac{1}{2}\cdot\log_2 P(y=e|x=1)$$
$$= -(1-p)\log_2(1-p) - p\log_2 p$$

- $C = H(Y) - H(Y|X)$

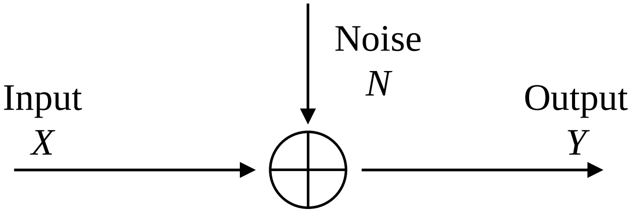
$$= 1 - p \text{ bits/symbol}$$

§ 2.3 Binary Erasure Channel (BEC)

- $C = 1 - p$ bits/symbol



§ 2.4 Additive White Gaussian Noise (AWGN) Channel



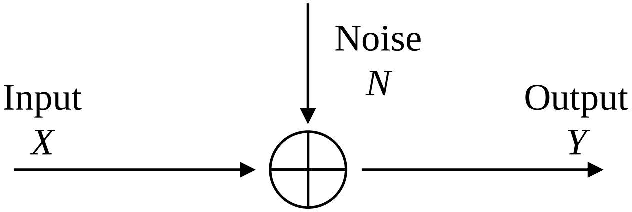
- Channel model $y_i = x_i + n_i$

  x_i: discrete input signal, a modulated signal

  n_i: white Gaussian noise as $\mathcal{N}(0, \sigma_N^2)$, independent of x_i

  y_i: continuous output signal, a variation of x_i
- It is a more realistic wireless channel model where the transmitted signal is impaired by noise.
- It is adopted to represent the space communication channel where light-of-sight (LoS) transmission is always ensured.
- It is also often used as a common platform for channel code comparison.

§ 2.4 Additive White Gaussian Noise (AWGN) Channel



- Channel model $y_i = x_i + n_i$
- Mutual Information: $I(X,Y) = H(Y) - H(Y|X)$

$$= H(Y) - H(X+N|X)$$

$$= H(Y) - H(N|X)$$

$$= H(Y) - H(N)$$

- Capacity: $C = \max_{P(x)}\{I(X,Y)\}$

$$= \max_{P(x)}\{H(Y) - H(N)\}$$

§ 2.4 Additive White Gaussian Noise (AWGN) Channel

- For AWGN: $\mathcal{N}(0, \sigma_N^2)$. Its pdf is

$$P(n) = \frac{1}{\sqrt{2\pi}\sigma_N} \exp\left(-\frac{n^2}{2\sigma_N^2}\right)$$

$$\begin{aligned} H(N) &= -\int_{-\infty}^{+\infty} P(n) \log_2 P(n) \mathrm{d}n \\ &= -\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma_N} \exp\left(-\frac{n^2}{2\sigma_N^2}\right) \log_2\left(\frac{1}{\sqrt{2\pi}\sigma_N} \exp\left(-\frac{n^2}{2\sigma_N^2}\right)\right) \mathrm{d}n \\ &= \frac{1}{2} \log_2(2\pi e \sigma_N^2) \quad \text{bits/symbol} \end{aligned}$$

- If input X is normal distributed (continuous) as $\mathcal{N}(\mu_X, \sigma_X^2)$, $I(X, Y)$ will be maximized and

$$C = H(Y) - H(N)$$

§ 2.4 Additive White Gaussian Noise (AWGN) Channel

- For Input: $\mathcal{N}(\mu_X, \sigma_X^2)$. Its pdf is

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left(-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right)$$

$$\begin{aligned} H(X) &= -\int_{-\infty}^{+\infty} P(x)\log_2 P(x)\mathrm{d}x \\ &= -\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left(-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right) \log_2\left(\frac{1}{\sqrt{2\pi}\sigma_X} \exp\left(-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right)\right) \mathrm{d}x \\ &= \frac{1}{2}\log_2(2\pi e\sigma_X^2) \quad \text{bits/symbol} \end{aligned}$$

- Since $Y = X + N$ and X and N are independent

  Output: $\mathcal{N}(\mu_X, \sigma_X^2 + \sigma_N^2) = \mathcal{N}(\mu_X, \sigma_Y^2)$

$$H(Y) = \frac{1}{2}\log_2(2\pi e(\sigma_X^2 + \sigma_N^2)) \quad \text{bits/symbol}$$

§ 2.4 Additive White Gaussian Noise (AWGN) Channel

- Channel model: $y_i = x_i + n_i$
- Capacity: $C = H(Y) - H(N)$

$$= \frac{1}{2}\log_2\left(2\pi e(\sigma_X^2 + \sigma_N^2)\right) - \frac{1}{2}\log_2(2\pi e\sigma_N^2)$$

$$= \frac{1}{2}\log_2\left(1 + \frac{\sigma_X^2}{\sigma_N^2}\right) \text{ bits/symbol}$$

- σ_X^2 is the power of the transmitted signal, while σ_N^2 is the power of noise. Hence, $\frac{\sigma_X^2}{\sigma_N^2}$ is often defined as the signal-to-noise ratio (SNR).
- This only defines the inachievable transmission limit since in practice, *X* will not be normal distributed.

§ 2.4 Additive White Gaussian Noise (AWGN) Channel

- **Band Limited AWGN Channel**
- In a practical system, sampling is needed at the receiver to reconstruct the received signal as Fig. 1.

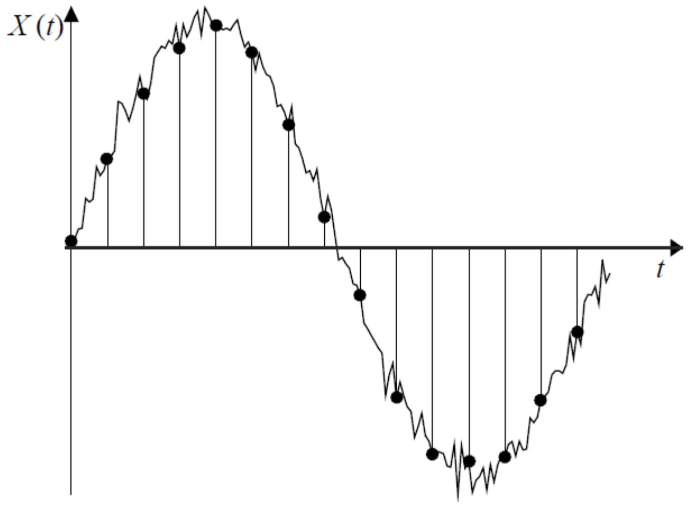


Fig. 1 Received Signal and Sampling

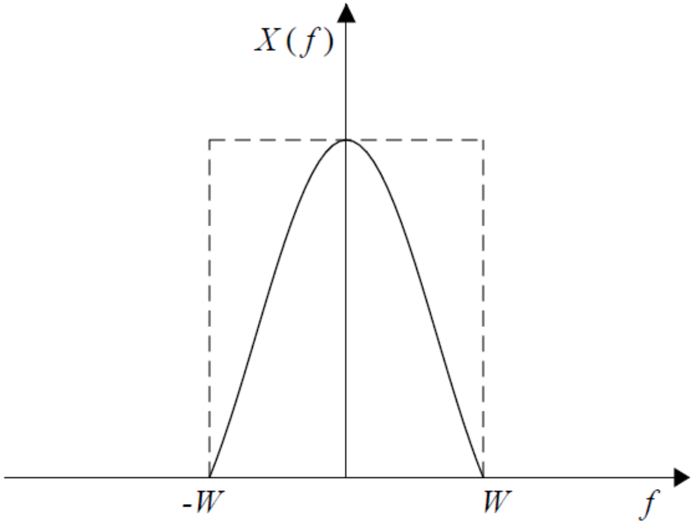


Fig. 2 Signal Sampling in frequency domain

- If the signal has a frequency of W, the sampling frequency should be at least $2W$ for perfect signal reconstruction. (Fig. 2)

§ 2.4 Additive White Gaussian Noise (AWGN) Channel

- **Band Limited AWGN Channel**
- With the sampling, we now have a series of time discrete Gaussian samples and the channel model becomes

$$y\left(t = \frac{s}{2W}\right) = x\left(t = \frac{s}{2W}\right) + n\left(t = \frac{s}{2W}\right), s = 1, 2, \cdots$$

- Signal $x\left(t = \frac{s}{2W}\right)$ has variance σ_X^2

  Noise $n\left(t = \frac{s}{2W}\right)$ has variance $\frac{N_0}{2}$, where N_0 is the noise power

- Capacity for each time discrete Gaussian channel

$$C_s = \frac{1}{2}\log_2\left(1 + \frac{2\sigma_X^2}{N_0}\right) \text{ bits/symbol}$$

§ 2.4 Additive White Gaussian Noise (AWGN) Channel

- **Band Limited AWGN Channel**
- Capacity of this band limited AWGN channel can be determined by

$$C = \frac{\sum_{s=1}^{2WT} C_s}{T}, \; T\text{–sampling duration}$$

- Since the average signal power

$$E = \frac{2WT \cdot \sigma_X^2}{T} = 2W\sigma_X^2$$

$$C_s = \frac{1}{2}\log_2\left(1 + \frac{E}{WN_0}\right) \text{ bits/symbol}$$

- Capacity of band limited AWGN channel becomes

$$C = \frac{2WT \cdot \frac{1}{2}\log_2\left(1 + \frac{E}{WN_0}\right)}{T} = W\log_2\left(1 + \frac{E}{WN_0}\right) \text{ bits/second}$$

§ 2.4 Additive White Gaussian Noise (AWGN) Channel

- **Shannon Limit:** Error free transmission over the Gaussian channel is possible if the signal-to-noise ratio $\frac{E_b}{N_0}$ is at least -1.6 dB.

Proof:

- This possibility is sealed by the use of channel code (information length *k* bits, codeword length *n* bits).
- Let E_b and E_c denote the energy of each information bit and each coded bit, respectively. It is required

$$k \cdot E_b = n \cdot E_c$$

so that adding redundancy does not increase the transmission energy.

- Consider each coded bit is carried by a modulated signal, e.g., using binary phase shift keying (BPSK),

$$E = E_c = \frac{E_b \cdot k}{n} = E_b \cdot r$$

§ 2.4 Additive White Gaussian Noise (AWGN) Channel

Continue the Proof

- Assume the signal frequency $W \to \infty$

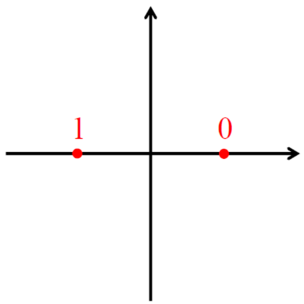
$$C = \lim_{W \to \infty} W \log_2 \left(1 + \frac{E}{N_0 W} \right)$$
$$= \frac{E}{N_0 \ln 2}$$
$$= \frac{E_b \cdot r}{N_0 \ln 2} \text{ bits/second}$$

- For error free transmission, it is required

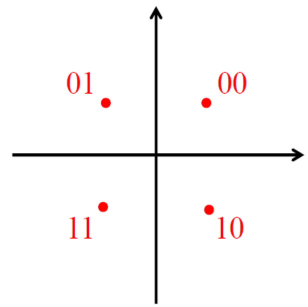
$$r < C \Longrightarrow \frac{E_b}{N_0} > \ln 2 = 0.69 = -1.6\text{dB}$$

§ 2.4 Additive White Gaussian Noise (AWGN) Channel

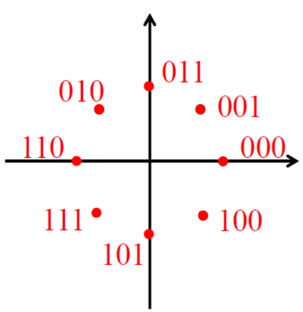
- **Finite Modulation Alphabets**
- In a wireless communication system, digital signal is modulated (mapped) to an analog signal for transmission.
- Commonly used modulation schemes include:



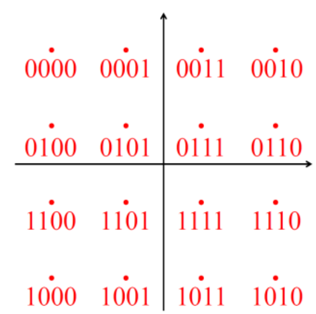
BPSK



QPSK



8PSK



16QAM

§ 2.4 Additive White Gaussian Noise (AWGN) Channel

- **Finite Modulation Alphabets**
- Input $X \in \{x_1, x_2, \ldots, x_M\}$, e.g., BPSK $M = 2$, QPSK $M = 4$, 8PSK $M = 8$, 16QAM $M = 16$, … .
- Channel Capacity

$$C = \max_{P(x)} \left\{ \sum_{i=1}^{M} \int_{y:-\infty}^{+\infty} P(x_i, y) \log_2 \frac{P(x_i|y)}{P(x_i)} \mathrm{d}y \right\}$$

Since

$$P(x_i, y) = P(y|x_i)P(x_i)$$

$$P(x_i|y) = \frac{P(y|x_i)P(x_i)}{P(y)}$$

$$P(y) = \sum_{i'=1}^{M} P(y|x_{i'})P(x_{i'})$$

§ 2.4 Additive White Gaussian Noise (AWGN) Channel

- **Finite Modulation Alphabets**

$$C = \max_{P(x_i)} \left\{ \sum_{i=1}^{M} P(x_i) \int_{y:-\infty}^{+\infty} P(y|x_i) \log_2 \frac{P(y|x_i)}{\sum_{i'=1}^{M} P(x_{i'}) P(y|x_{i'})} \mathrm{d}y \right\}$$

- Assume each modulated symbol is equally likely to be transmitted

$$P(x_i) = P(x_{i'}) = \frac{1}{M}.$$

- Capacity:

$$C = \frac{1}{M} \sum_{i=1}^{M} \int_{y:-\infty}^{+\infty} P(y|x_i) \log_2 \frac{P(y|x_i)}{\frac{1}{M} \sum_{i'=1}^{M} P(y|x_{i'})} \mathrm{d}y \ \text{bits/symbol}$$

§ 2.4 Additive White Gaussian Noise (AWGN) Channel

- **Finite Modulation Alphabets**
- Over the AWGN Channel $y = x_i + n$

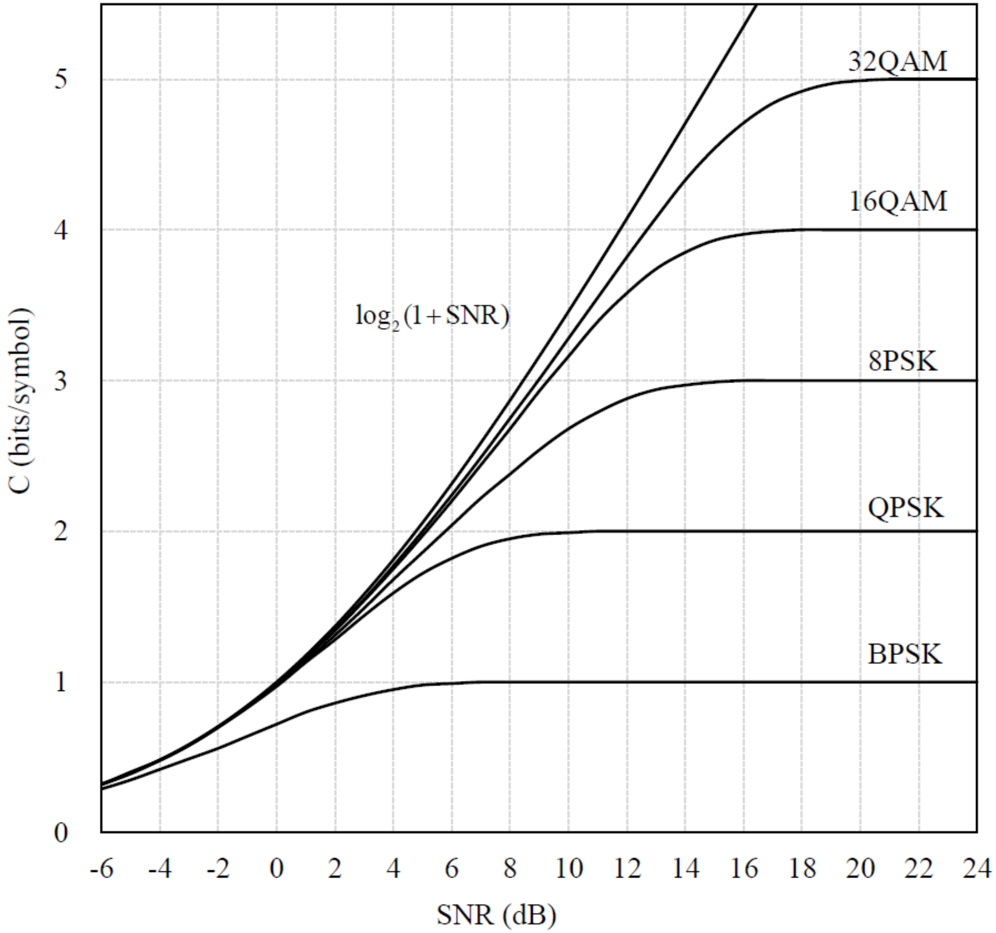
$$P(y|x_i) = \frac{1}{\sqrt{2\pi}\sigma_N} \exp\left(-\frac{|y - x_i|^2}{2\sigma_N^2}\right)$$
$$= \frac{1}{\sqrt{2\pi}\sigma_N} \exp\left(-\frac{|n|^2}{2\sigma_N^2}\right)$$

- Capacity:

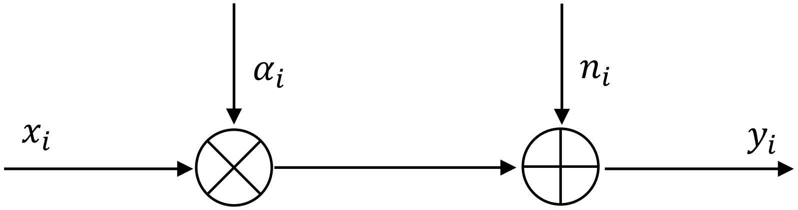
$$C = \frac{1}{M}\sum_{i=1}^{M} \mathbb{E}\left[\log_2 \frac{P(y|x_i)}{\frac{1}{M}\sum_{i'=1}^{M} P(y|x_{i'})}\right]$$
$$= \log_2 M - \frac{1}{M}\sum_{i=1}^{M} \mathbb{E}\left[\log_2 \sum_{i'=1}^{M} \exp\left(-\frac{|x_i + n - x_{i'}|^2 - |n|^2}{2\sigma_N^2}\right)\right] \text{ bits/symbol}$$

§ 2.4 Additive White Gaussian Noise (AWGN) Channel

- **Finite Modulation Alphabets**



§ 2.5 Fading Channels



- Channel Model: $y_i = \alpha_i \cdot x_i + n_i$

  If α_i is Rayleigh distributed following $\alpha_i = |\alpha_i|e^{j\varphi_i}$, $P(|\alpha_i|) = 2|\alpha_i|e^{-|\alpha_i|^2}$ and $P(\varphi_i) = \frac{1}{2\pi}\text{rect}\left(\frac{\varphi_i}{2\pi}\right)$. It is called the Rayleigh fading channel.
- Fading coefficients α_i further represent the effect of signal attenuation, signal scattering, path loss and multi-path accumulation.
- It is a channel model often used for urban communications.
- Fading types:

  (1) Fast fading: α_i changes independently for every x_i.

  (2) Quasi-static fading: α_i remains unchanged during the transmission of a codeword and changes independently from codeword to codeword.

  (3) Block fading: α_i changes independently block by block.

§ 2.5 Fading Channels

- Assume α_i are known by both the transmitter and receiver.
- Instantaneous capacity:

$$C(\alpha_i) = W\log_2\left(1 + \frac{\alpha_i^2 \cdot P(\alpha_i)}{WN_0}\right)$$

$P(\alpha_i)$: the signal power depending on α_i.

It is the maximal achievable transmission rate defined by a particular fading coefficient α_i.

- Ergodic Capacity:

$$C = \max_{P(\alpha_i)} \mathbb{E}\left[W\log_2\left(1 + \frac{\alpha_i^2 \cdot P(\alpha_i)}{WN_0}\right)\right]$$

It is the average transmission rate that can be realized over all channel states.

References:

[1] Elements of Information Theory, by T. Cover and J. Thomas.

[2] Scriptum for the lectures, Applied Information Theory, by M. Bossert.