Chapter 5 Low-Density Parity-Check Codes

- 5.1 Introduction of LDPC Codes
- 5.2 Tanner Graph Representation
- 5.3 Encoding of LDPC Code
- 5.4 Belief Propagation Decoding
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§ 5.1 Introduction of LDPC Code

- Introduction
  - Proposed by Robert Gallager in 1962 [1].
  - It was overlooked for over three decades until 1995, it was rediscovered by David Mackay [2].
  - It is a linear block code defined by its sparse parity-check matrix which is inherently good for the belief propagation decoding.
  - It can well approach the Shannon capacity with a decoding complexity that is quadratic in the dimension of the code.
  - Its potential applications include wireless communications and storage devices.

[1] R. Gallager, "Low-Density Parity-Check Codes," *IRE Trans. Inform. Theory*, vol. IT-8, pp21-28, Jan, 1962.

[2] D. Mackay and R. Neal, "Good codes based on very sparse matrices", in *the 5th IMA Conf. Cryptography and Coding*, lecture notes in Computer Science Springer. 1995.

§ 5.1 Introduction of LDPC Codes

– LDPC code: A linear block code whose parity-check matrix **H** has sparse non-zero elements. For a binary LDPC code, its matrix **H** has sparse 1s.

– Column weight (w_c): Number of 1s in a column of **H.**
 Row weight (w_r): Number of 1s in a row of **H.**

– Regular LDPC codes: Each column of **H** has the same column weight, and each row of the **H** has the same row weight. It is normally denoted as a (w_c , w_r , N) LDPC code, where N is the codeword length.

– Irregular LDPC codes: The parity-check matrix has varying column weights and row weights.

– In general, irregular codes have better performance than regular codes. But irregular codes are more difficult to implement.

§ 5.1 Introduction of LDPC Codes

Example 5.1 A regular LDPC code has a parity-check matrix of

$$\mathbf{H} = \begin{array}{c} \begin{array}{cccccccccc} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{array}$$

$w_c = 3$, $w_r = 6$, $M = 5$, $N = 10$.

M : Number of parity-check equations. The above matrix implies

$$z_1 : c_1 + c_2 + c_3 + c_6 + c_7 + c_{10} = 0$$
$$z_2 : c_1 + c_3 + c_5 + c_6 + c_8 + c_9 = 0$$
$$z_3 : c_3 + c_4 + c_5 + c_7 + c_9 + c_{10} = 0$$
$$z_4 : c_2 + c_4 + c_5 + c_6 + c_8 + c_{10} = 0$$
$$z_5 : c_1 + c_2 + c_4 + c_7 + c_8 + c_9 = 0$$

- If all rows of **H** are independent, $M = N - K$. Otherwise $M > N - K$.
- Uniform row weight requires $\frac{w_r}{N} = \frac{w_c}{M}$. If $M = N - K$, then the code rate is $R = \frac{K}{N} = 1 - \frac{M}{N} = 1 - \frac{w_c}{w_r}$. If $M > N - K$, $R > 1 - \frac{w_c}{w_r}$.

actual code rate **design code rate**

§ 5.1 Introduction of LDPC Codes

Example 5.2 Construct a (3, 4, 20) regular LDPC code.

Given a based matrix **A** as：

$$\mathbf{A} = \begin{bmatrix} 1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{bmatrix}$$

Let $\pi_i(\mathbf{A})$ denote a random permutation function that permutes the columns of **A**.

§ 5.1 Introduction of LDPC Codes

The patiry-check matrix of the (3, 4, 20) regular LDPC code can be generated by

$$
\mathbf{H} = \begin{bmatrix} \mathbf{A} \\ \pi_1(\mathbf{A}) \\ \pi_2(\mathbf{A}) \end{bmatrix} = \left[\begin{array}{cccccccccccccccccccc}
1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1\\
\hline
1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0\\
0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&1&0&0&0\\
0&0&1&0&0&0&1&0&0&0&0&0&0&1&0&0&0&1&0&0\\
0&0&0&1&0&0&0&0&0&0&1&0&0&0&1&0&0&0&1&0\\
0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1\\
\hline
1&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0\\
0&1&0&0&0&0&1&0&0&0&1&0&0&0&0&1&0&0&0&0\\
0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&1&0\\
0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&1&0&0&0\\
0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1
\end{array}\right]
$$

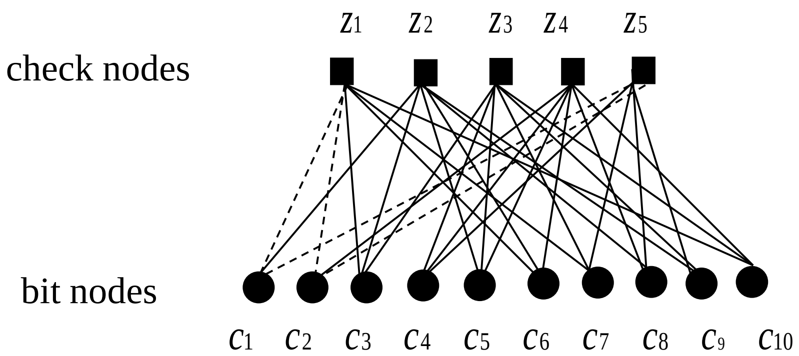
Since there are 13 independent rows, the code's dimension is $K = 20 - 13 = 7$.

The rate of the code is $R = 0.35 > 1 - \frac{w_c}{w_r}$.

Q: Why is random permutation of columns of* A *necessary ?

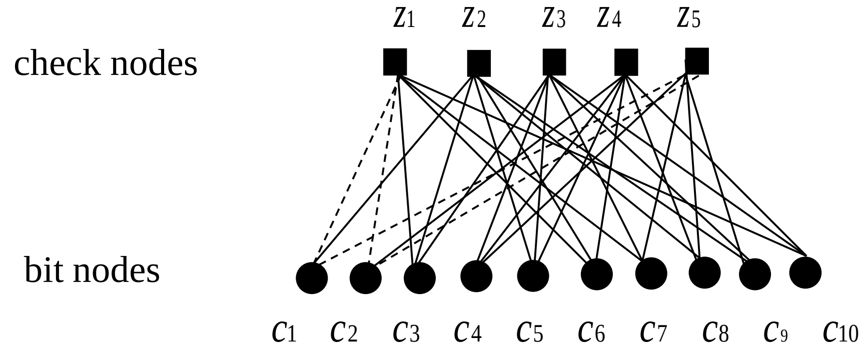
§ 5.2 Tanner Graph Representation

- The parity-check matrix **H** [h_{mn}] can be represented as a Tanner graph.
- The parity-check matrix **H** of **Example 5.1** can be shown as :



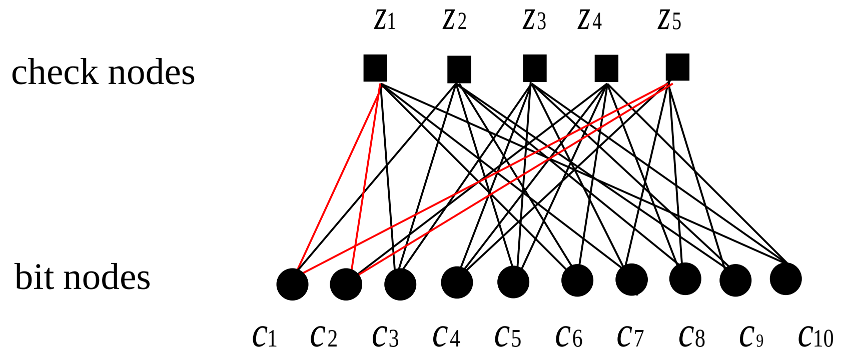
- The Tanner graph has two sets of nodes, the check nodes (z_m) and the bit nodes (c_n). There is a connection between z_m and c_n if h_{mn}= 1.
- Belief propagation decoding of a LDPC code is performed based on a Tanner graph : propagating soft information between the check nodes and the bit nodes through the established connections.

§ 5.2 Tanner Graph Representation



- $N_m = \{n : h_{mn} = 1\}$ — The set of bits that participate the check z_m.
  E.g., $N_1 = \{1, 2, 3, 6, 7, 10\}$, $N_3=\{3, 4, 5, 7, 9, 10\}$.
- $N_{m\backslash n}$ — The set of bits except c_n that participate check z_m.
  E.g., $N_{1\backslash 3} = \{1, 2, 6, 7, 10\}$.
- $M_n = \{m : h_{mn} = 1\}$ — The set of checks in which bit c_n is involved.
  E.g., $M_1 = \{1, 2, 5\}$, $M_{10} = \{1, 3, 4\}$.
- $M_{n\backslash m}$ — The set of checks except check z_m in which bit c_n is involved.
  E.g., $M_{1\backslash 2} = \{1, 5\}$.

§ 5.2 Tanner Graph Representation



- For a regular LDPC code, every check node is connected to $|N_m|$ bit nodes where $|N_m| = w_r$, and every bit node is connected to $|M_n|$ check nodes where $|M_n| = w_c$.

- Girth : the shortest cycle in a Tanner graph and it is ≥ 4. It is desirable to avoid a LDPC code whose Tanner graph has a girth of 4 as it would degrade the decoding performance. (In the above Tanner graph, the highlighted cycle is of length 4 and hence the LDPC code has a girth of 4.)

§ 5.3 Encoding of LDPC Codes

– By performing Gaussian elimination, a parity-check matrix $\mathbf{H}$ can be transformed into

$$\mathbf{H} = [\, \mathbf{I}_M \,\vdots\, \mathbf{P} \,]$$

where $\mathbf{I}_M$ is a $M \times M$ identity matrix.

– Its corresponding generator matrix $\mathbf{G}$ can be written as :

$$\mathbf{G} = [\, \mathbf{P}^T \,\vdots\, \mathbf{I}_K]$$

where $\mathbf{I}_K$ is a $K \times K$ identity matrix.

– Encoding of a K dimensional message vector $\bar{m} = [m_1, m_2, \dots, m_K]$ is done by

$$\begin{aligned} \bar{c} &= \bar{m} \cdot \mathbf{G} \\ &= [c_1, c_2, \dots, c_{N-K}, c_{N-K+1}, \dots, c_N] \\ &= [p_1, p_2, \dots, p_{N-K}, m_1, \dots, m_K]. \end{aligned}$$

§ 5.3 Encoding of LDPC Codes

Example 5.3 By performing Gaussian elimination on the matrix **H** of **Example 5.1**, we have

$$\mathbf{H} = \left[\begin{array}{ccccc:ccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right]$$

Hence, the generator matrix **G** is

$$\mathbf{G} = \left[\begin{array}{ccccc:ccccc} 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

If the message vector is $\bar{m} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \end{bmatrix}$, the codeword $\bar{c}$ is generated as

$$\bar{c} = \bar{m} \cdot \mathbf{G} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

§ 5.4 Belief Propagation Decoding

– Belief Propagation (BP) decoding is performed based on the Tanner graph of the LDPC code.

– Optimal decoding estimates a codeword by maximizing

$$\Pr[\bar{c} \mid z_m = 0, \forall m]$$

Its complexity is $O(2^K)$.

– Suboptimal decoding estimates individual coded bit c_n by maximizing

$$\Pr[c_n = \theta \mid z_m = 0, m \in M_n], \theta \in \{0,1\}$$

Its complexity is $O(K^2)$.

– BP decoding is a sub-optimal decoding algorithm.

§ 5.4 Belief Propagation Decoding

– BP decoding is to update the following two probabilities iteratively.

1. The probability of bit $c_n = \theta$ ($\theta \in \{0,1\}$) conditioned on all its associated checks except z_m are satisfied, i.e.,

$$q_{mn}(\theta) = \Pr[c_n = \theta \mid z_{m'} = 0, m' \in M_{n \setminus m}] \quad (1)$$

2. The probability of check z_m is satisfied conditioned on bit $c_n = \theta$, i.e.,

$$r_{mn}(\theta) = \Pr[z_m = 0 \mid c_n = \theta] \quad (2)$$

§ 5.4 Belief Propagation Decoding

– Since there are N coded bits and M checks, $q_{mn}(\theta)$ and $r_{mn}(\theta)$ should be accommodated in matrices **Q** and **R**, respectively. **Q** and **R** are of size $2M \times N$.

$$\mathbf{Q} = \left[\begin{array}{c:c:cc:c} q_{11}(0) & q_{12}(0) & \cdots & \cdots & q_{1N}(0) \\ q_{11}(1) & q_{12}(1) & \cdots & \cdots & q_{1N}(1) \\ \hdashline q_{21}(0) & q_{22}(0) & \cdots & \cdots & q_{2N}(0) \\ q_{21}(1) & q_{22}(1) & \cdots & \cdots & q_{2N}(1) \\ \hdashline \vdots & \vdots & \cdots & \cdots & \vdots \\ \hdashline q_{M1}(0) & q_{M2}(0) & \cdots & \cdots & q_{MN}(0) \\ q_{M1}(1) & q_{M2}(1) & \cdots & \cdots & q_{MN}(1) \end{array}\right] \quad \mathbf{R} = \left[\begin{array}{c:c:cc:c} r_{11}(0) & r_{12}(0) & \cdots & \cdots & r_{1N}(0) \\ r_{11}(1) & r_{12}(1) & \cdots & \cdots & r_{1N}(1) \\ \hdashline r_{21}(0) & r_{22}(0) & \cdots & \cdots & r_{2N}(0) \\ r_{21}(1) & r_{22}(1) & \cdots & \cdots & r_{2N}(1) \\ \hdashline \vdots & \vdots & \cdots & \cdots & \vdots \\ \hdashline r_{M1}(0) & r_{M2}(0) & \cdots & \cdots & r_{MN}(0) \\ r_{M1}(1) & r_{M2}(1) & \cdots & \cdots & r_{MN}(1) \end{array}\right]$$

– BP decoding iterations $\mathbf{Q} \underset{\text{Vertical update}}{\overset{\text{Horizontal update}}{\rightleftarrows}} \mathbf{R}$.

– After a number of iterations, the decision on all the bits c_n is made based on **Q** by

$$q_n(\theta) = \Pr[c_n = \theta \mid z_m = 0, m \in M_n] \qquad (3)$$

§ 5.4 Belief Propagation Decoding

– **Initialization:**

Given a received symbol vector $\bar{y} = (y_1, y_2, \dots, y_N)$, one could obtain the channel observations for all the coded bits as

$$\begin{cases} f_1(0) = \Pr(y_1|c_1 = 0) \\ f_1(1) = \Pr(y_1|c_1 = 1) \end{cases}, \begin{cases} f_2(0) = \Pr(y_2|c_2 = 0) \\ f_2(1) = \Pr(y_2|c_2 = 1) \end{cases}, \dots, \begin{cases} f_N(0) = \Pr(y_N|c_N = 0) \\ f_N(1) = \Pr(y_N|c_N = 1) \end{cases}$$

Before decoding, we assume $\Pr(c_n = 0) = \Pr(c_n = 1) = \frac{1}{2}, \ \forall n.$

Hence,

$$\Pr(c_n = \theta|y_n) = \Pr(y_n|c_n = \theta) = f_n(\theta), \theta \in \{0,1\}, \forall\, n.$$

Matrix **Q** is initialized by

$$q_{mn}(\theta) = f_n(\theta) \cdot h_{mn}, \ \ \forall m, n.$$

§ 5.4 Belief Propagation Decoding

– **Horizontal update:** update $\mathbf{R}\big(r_{mn}(\theta)\big)$ by $\mathbf{Q}\big(q_{mn}(\theta)\big)$.

$$
\begin{aligned}
r_{mn}(\theta) &= \Pr[z_m = 0 \mid c_n = \theta] \\
&= \sum_{\{c_{n'}, \Sigma\theta_{n'} = \theta\}} \Pr[z_m = 0, \{c_{n'} = \theta_{n'}, n' \in N_m \backslash n\} \mid c_n = \theta] \\
&= \sum_{\{c_{n'}, \Sigma\theta_{n'} = \theta\}} \Pr[z_m = 0 \mid \{c_{n'} = \theta_{n'}, n' \in N_m \backslash n\}, c_n = \theta] \cdot \Pr[\{c_{n'} = \theta_{n'}, n' \in N_m \backslash n\}] \\
&= \sum_{\{c_{n'}, \Sigma\theta_{n'} = \theta\}} \Pr[z_m = 0 \mid \{c_{n'} = \theta_{n'}, n' \in N_m \backslash n\}, c_n = \theta] \cdot \prod_{n' \in N_m \backslash n} \Pr(c_{n'} = \theta_{n'}) \qquad (4) \\
&= \sum_{\{c_{n'}, \Sigma\theta_{n'} = \theta\}} \prod_{n' \in N_m \backslash n} \boxed{\Pr(c_{n'} = \theta_{n'})} \longleftarrow q_{mn'}(\theta)
\end{aligned}
$$

Remark: In (4), it is assumed that all codes bits c_n are independent. Moreover, for $\{c_{n'},\ n' \in N_m \backslash n\}$, if $\Sigma\theta_{n'} = \theta$, $\Pr[z_m = 0 \mid \{c_{n'} = \theta_{n'}, n' \in N_m \backslash n\}, c_n = \theta] = 1$. Otherwise, $\Pr[z_m = 0 \mid \{c_{n'} = \theta_{n'}, n' \in N_m \backslash n\}, c_n = \theta] = 0$.

§ 5.4 Belief Propagation Decoding

– **Horizontal update:** update **R** by **Q**.

$$r_{mn}(\theta) = \sum_{\theta=\sum_{n'\in N_m\backslash n}\theta_{n'}} \prod_{n'\in N_m\backslash n} q_{mn'}(\theta)$$

With $c_n = \theta$, $\theta = \Sigma\theta_{n'}$ for $n' \in N_m\backslash n$ ensures check z_m is satisfied, i.e.,

$$z_m = \sum_{n\in N_m} c_n = \theta + \sum_{n'\in N_m\backslash n} \theta_{n'} = 0$$

– **Example 5.4** For the LDPC code of **Example 5.1**, if we want to update $r_{11}(1) = \Pr[z_1 = 0 \mid c_1 = 1]$, we need the remaining bits of z_1 satisfy $c_2+c_3+c_6+c_7+c_{10}=1$. Bits $c_2\, c_3\, c_6\, c_7\, c_{10}$ have the following 16 permutations:

10000, 01000, 00100, 00010, 00001, 11100, 01110, 00111,
11001, 11010, 01101, 10101, 10011, 01011, 10110, 11111.

Hence, $r_{11}(1)$ is updated by summing the following 16 products.

$$\left.\begin{matrix} q_{12}(1)q_{13}(0)q_{16}(0)q_{17}(0)q_{10}(0) \\ \vdots \\ q_{12}(1)q_{13}(1)q_{16}(1)q_{17}(1)q_{10}(1) \end{matrix}\right\} 16$$

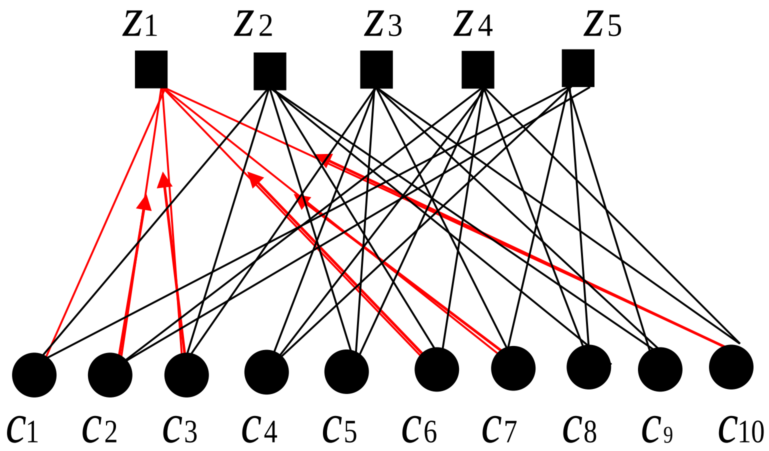
§ 5.4 Belief Propagation Decoding

– **Horizontal update:** update **R** by **Q**

$$r_{mn}(\theta) = \sum_{\theta=\sum_{n'\in N_m\setminus n}\theta_{n'}} \prod_{n'\in N_m\setminus n} q_{mn'}(\theta)$$

– Tanner graph reflection.

- The update of $r_{11}(1)$ of **Example 5.4** can be seen as



$r_{11}(1) = \Pr[z_1 = 0|c_1 = 1]$.

The red edges provide information to calculate probability of the black edge.

§ 5.4 Belief Propagation Decoding

- **Vertical update:** update $\mathbf{Q}\big(q_{mn}(\theta)\big)$ by $\mathbf{R}(r_{mn}(\theta))$.

$$q_{mn}(\theta) = \Pr[c_n = \theta \mid z_{m'} = 0, m' \in M_n \backslash m]$$

$$= \frac{\Pr[z_{m'} = 0, m' \in M_n \backslash m \mid c_n = \theta] \cdot \Pr[c_n = \theta]}{\Pr[z_{m'} = 0, m' \in M_n \backslash m]}$$

$$= \frac{\prod_{m' \in M_n \backslash m} \Pr[z_{m'} = 0 \mid c_n = \theta] \cdot \Pr[c_n = \theta]}{\Pr[z_{m'} = 0, m' \in M_n \backslash m]} \quad (5)$$

$$= \alpha_{mn} \cdot \prod_{m' \in M_n \backslash m} \underbrace{\Pr[z_{m'} = 0 \mid c_n = \theta]}_{r_{m'n(\theta)}} \cdot \underbrace{\Pr[c_n = \theta]}_{f_n(\theta)} \quad (6)$$

- In (5), it is assumed that all cheeks are independent.
- In (6), α_{mn} is a normalization factor that ensures $q_{mn}(0) + q_{mn}(1) = 1$.

§ 5.4 Belief Propagation Decoding

– **Vertical update:** update **Q** by **R**.

$$q_{mn}(\theta) = \alpha_{mn} \cdot f_n(\theta) \cdot \prod_{m' \in M_n \backslash m} r_{m'n}(\theta)$$

α_{mn} is a normalization factor that ensures $q_{mn}(0) + q_{mn}(1) = 1$, i.e.,

$$\alpha_{mn} = \left[\sum_{\theta \in \{0,1\}} f_n(\theta) \cdot \prod_{m' \in M_n \backslash m} r_{m'n}(\theta) \right]^{-1}$$

– **Example 5.5** (Continue from **Example 5.4**), if we want to apdate

$$q_{11}(\theta) = \Pr[c_1 = \theta \mid z_{m'} = 0, m' \in M_{1\backslash 1}]$$

we need to calculate

$$q_{11}(0) = \alpha_{11} \cdot f_1(0) \cdot (r_{21}(0) \times r_{51}(0))$$
$$q_{11}(1) = \alpha_{11} \cdot f_1(1) \cdot (r_{21}(1) \times r_{51}(1))$$

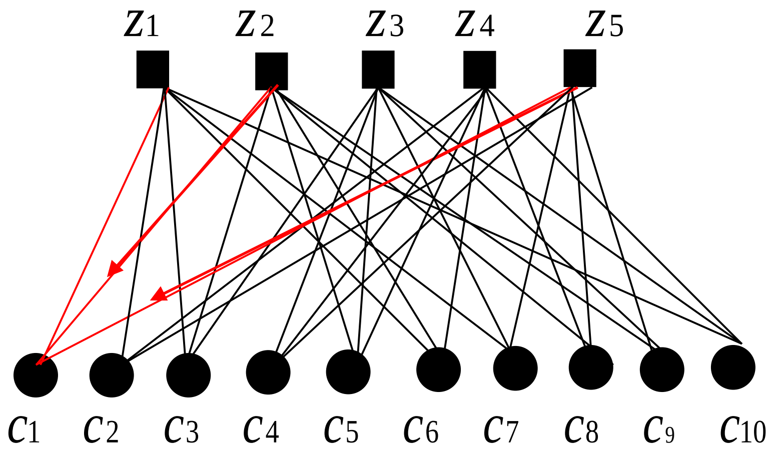
§ 5.4 Belief Propagation Decoding

– **Vertical update:** update **Q** by **R**

$$q_{mn}(\theta) = \alpha_{mn} \cdot f_n(\theta) \cdot \prod_{m' \in M_n \backslash m} r_{m'n}(\theta)$$

– Tanner graph reflection.

- The update of $q_{11}(\theta)$ of **Example 5.5** can be seen as



$q_{11}(1) = f_1(1) \cdot r_{21}(1) \cdot r_{51}(1)$

Again, the red edges provide information to update probability of the black edge.

§ 5.4 Belief Propagation Decoding

- After each horizontal-vertical iteration, we can calculate $q_n(\theta)$ of (3) by

$$q_n(\theta) = \alpha_n \cdot f_n(\theta) \cdot \prod_{m \in M_n} r_{mn}(\theta)$$

α_n is a normalization factor that ensures $q_n(0) + q_n(1) = 1$.

$$\alpha_n = \left[\sum_{\theta \in \{0,1\}} f_n(\theta) \cdot \prod_{m \in Mn} r_{mn}(\theta) \right]^{-1}$$

- Decision on bit c_n

$$\begin{cases} c_n = 0, & if \quad q_n(0) > q_n(1) \\ c_n = 1, & if \quad q_n(0) < q_n(1) \end{cases}$$

- After decisions are made on all the coded bits, we can obtain an estimated codeword $\hat{c}$. The iteration will be terminated if $\hat{c}$ is a valid codeword, i.e., $\hat{c} \cdot \mathbf{H}^{\mathrm{T}} = 0$. Otherwise, the iterative horizontal-vertical updates continue until $\hat{c} \cdot \mathbf{H}^{\mathrm{T}} = 0$ is satisfied, or the designed maximal iteration number is reached.
- The BP decoding algorithm is also called the **Sum-Product algorithm**.

§ 5.4 Belief Propagation Decoding

Why low density of H is important for BP decoding ?

- The Horizontal update computation of $\prod_{n' \in N_{m \backslash n}} q_{mn'}(\theta)$ assumes that all the coded bits are independent. Similarly, the Vertical update of $\prod_{m' \in M_{n \backslash m}} r_{m'n}(\theta)$ assumes all checks are independent.

- However, once cycles exist in the Tanner graph, the independence will disappear. For example, when two coded bits are involved in the same two checks, a cycle of length 4 will exist in the Tanner graph.

- A low density **H** inherits less cycles especially the cycles of length 4. The BP decoding would favour this type of code — low-density parity-check codes.

§ 5.4 Belief Propagation Decoding

Why low density of H is important for BP decoding ?

Example 5.6 Let us look at BP decoding of the LDPC code of **Example 5.1**.

– By examing the Tanner graph, we can see coded bits c_1 and c_2 are involved in both checks z_1 and z_5, yielding a cycle of length 4.

– Horizontal update :

$r_{11}(\theta) \longleftarrow q_{12}(\theta)$ ③, $q_{13}(\theta), q_{16}(\theta), q_{17}(\theta), q_{1,10}(\theta)$

$r_{12}(\theta) \longleftarrow q_{11}(\theta)$ ③, $q_{13}(\theta), q_{16}(\theta), q_{17}(\theta), q_{1,10}(\theta)$

:

$r_{51}(\theta) \longleftarrow q_{52}(\theta)$ ①, $q_{54}(\theta), q_{57}(\theta), q_{58}(\theta), q_{59}(\theta)$

$r_{52}(\theta) \longleftarrow q_{51}(\theta)$ ①, $q_{54}(\theta), q_{57}(\theta), q_{58}(\theta), q_{59}(\theta)$

– Vertical update :

$q_{11}(\theta) \longleftarrow r_{21}(\theta), r_{51}(\theta)$ ②

$q_{12}(\theta) \longleftarrow r_{42}(\theta), r_{52}(\theta)$ ②

– Observations:

1) ① — ② process, bits c_1 and c_2 start to correlate.

2) ① — ② — ③ process, part of the information used to update $r_{mn}(\theta)$ comes for c_n itself.

§ 5.4 Belief Propagation Decoding

Example 5.7 (Continue from **Example 5.3**). If the LDPC codeword $\bar{c} = [0\quad 0\quad 0\quad 1\quad 0\quad 1\quad 0\quad 1\quad 0\quad 1]$ is transmitted to a memoryless channel, with the received symbol vector $\bar{y}$, we obtain the channel observation matrix **F** as

$$\mathbf{F} = \begin{bmatrix} 0.78 & 0.84 & 0.81 & 0.52 & 0.45 & 0.13 & 0.82 & 0.21 & 0.75 & 0.24 \\ 0.22 & 0.16 & 0.19 & 0.48 & 0.55 & 0.87 & 0.18 & 0.79 & 0.25 & 0.76 \end{bmatrix}$$

Matrix **Q** is initialized as :

$$\mathbf{Q} = \begin{bmatrix} 0.78 & 0.84 & 0.81 & 0 & 0 & 0.13 & 0.82 & 0 & 0 & 0.24 \\ 0.22 & 0.16 & 0.19 & 0 & 0 & 0.87 & 0.18 & 0 & 0 & 0.76 \\ 0.78 & 0 & 0.81 & 0 & 0.45 & 0.13 & 0 & 0.21 & 0.75 & 0 \\ 0.22 & 0 & 0.19 & 0 & 0.55 & 0.87 & 0 & 0.79 & 0.25 & 0 \\ 0 & 0 & 0.81 & 0.52 & 0.45 & 0 & 0.82 & 0 & 0.75 & 0 \\ 0 & 0 & 0.19 & 0.48 & 0.55 & 0 & 0.18 & 0 & 0.25 & 0.76 \\ 0 & 0.84 & 0 & 0.52 & 0.45 & 0.13 & 0 & 0.21 & 0 & 0.24 \\ 0 & 0.16 & 0 & 0.48 & 0.55 & 0.87 & 0 & 0.79 & 0 & 0.76 \\ 0.78 & 0.84 & 0 & 0.52 & 0 & 0 & 0.82 & 0.21 & 0.75 & 0 \\ 0.22 & 0.16 & 0 & 0.48 & 0 & 0 & 0.18 & 0.79 & 0.25 & 0 \end{bmatrix}$$

§ 5.4 Belief Propagation Decoding

After the 1st Horizontal-Vertical iteration, we have

$$\mathbf{R} = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c}
0.551914 & 0.542753 & 0.546890 & 0 & 0 & 0.460714 & 0.545425 & 0 & 0 & 0.444092 \\
0.448086 & 0.457247 & 0.453110 & 0 & 0 & 0.539286 & 0.454575 & 0 & 0 & 0.555908 \\
\hline
0.493347 & 0 & 0.493991 & 0 & 0.537255 & 0.505034 & 0 & 0.506423 & 0.493347 & 0 \\
0.506653 & 0 & 0.506009 & 0 & 0.462745 & 0.494966 & 0 & 0.493577 & 0.507451 & 0 \\
\hline
0 & 0 & 0.500333 & 0.505158 & 0.497937 & 0 & 0.500322 & 0 & 0.500413 & 0.499603 \\
0 & 0 & 0.499667 & 0.494842 & 0.502063 & 0 & 0.499678 & 0 & 0.499587 & 0.500397 \\
\hline
0 & 0.500446 & 0 & 0.507588 & 0.496965 & 0.499590 & 0 & 0.499477 & 0 & 0.499416 \\
0 & 0.499554 & 0 & 0.492412 & 0.503035 & 0.500410 & 0 & 0.500523 & 0 & 0.500584 \\
\hline
0.497476 & 0.497921 & 0 & 0.464662 & 0 & 0 & 0.497791 & 0.502437 & 0.497173 & 0 \\
0.502524 & 0.502079 & 0 & 0.535338 & 0 & 0 & 0.502209 & 0.497563 & 0.502827 & 0
\end{array}\right]$$

§ 5.4 Belief Propagation Decoding

$$\mathbf{Q} = \begin{bmatrix} 0.773636 & 0.839121 & 0.806481 & 0 & 0 & 0.132106 & 0.818884 & 0 & 0 & 0.239285 \\ 0.226364 & 0.160879 & 0.193519 & 0 & 0 & 0.867894 & 0.181116 & 0 & 0 & 0.760715 \\ 0.812140 & 0 & 0.837461 & 0 & 0.444958 & 0.113039 & 0 & 0.211273 & 0.748185 & 0 \\ 0.187860 & 0 & 0.162539 & 0 & 0.555042 & 0.886961 & 0 & 0.788727 & 0.251815 & 0 \\ 0 & 0 & 0.833978 & 0.492203 & 0.484126 & 0 & 0.844187 & 0 & 0.742212 & 0.201076 \\ 0 & 0 & 0.166022 & 0.507797 & 0.515874 & 0 & 0.155813 & 0 & 0.257788 & 0.798924 \\ 0 & 0.860727 & 0 & 0.489773 & 0.485097 & 0.115241 & 0 & 0.215940 & 0 & 0.201196 \\ 0 & 0.139273 & 0 & 0.5110227 & 0.514903 & 0.884759 & 0 & 0.784060 & 0 & 0.798804 \\ 0.809608 & 0.861934 & 0 & 0.532711 & 0 & 0 & 0.845514 & 0.213942 & 0.744684 & 0 \\ 0.190392 & 0.138066 & 0 & 0.467289 & 0 & 0 & 0.154486 & 0.786058 & 0.255316 & 0 \end{bmatrix}$$

Hence, the *a posteriori* probability matrix **Q'** is :

$$\mathbf{Q'} = \begin{bmatrix} 0.808046 & 0.860941 & 0.834162 & 0.497361 & 0.482065 & 0.115074 & 0.844356 & 0.215586 & 0.742528 & 0.200821 \\ 0.191954 & 0.139059 & 0.165838 & 0.502639 & 0.517935 & 0.884926 & 0.155644 & 0.784414 & 0.257472 & 0.799179 \end{bmatrix}$$

The estimated codeword is $\hat{c} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$. It does not satisfy $\hat{c} \cdot \mathbf{H}^{\mathrm{T}} = 0$ and the iteration continues...

§ 5.4 Belief Propagation Decoding

After the 3rd Horizontal-Vertical iteration, we have

R =

0.549960	0.540086	0.544369	0	0	0.463092	0.542650	0	0	0.447890
0.450040	0.459914	0.455631	0	0	0.536908	0.457350	0	0	0.552110
0.493114	0	0.493650	0	0.545393	0.505532	0	0.507453	0.491301	0
0.506886	0	0.506350	0	0.454607	0.494468	0	0.492547	0.508699	0
0	0	0.499989	0.500176	0.502649	0	0.499989	0	0.499985	0.500012
0	0	0.500011	0.499824	0.497351	0	0.500011	0	0.500015	0.499988
0	0.499975	0	0.500415	0.503915	0.500023	0	0.500032	0	0.500030
0	0.500025	0	0.499585	0.496085	0.49977	0	0.499968	0	0.499970
0.496595	0.497094	0	0.457904	0	0	0.496955	0.503693	0.495668	0
0.503405	0.502906	0	0.542096	0	0	0.503045	0.496307	0.504332	0

§ 5.4 Belief Propagation Decoding

$$\mathbf{Q} = \begin{bmatrix}
0.772854 & 0.838418 & 0.806053 & 0 & 0 & 0.132534 & 0.818189 & 0 & 0 & 0.240031 \\
0.227146 & 0.161582 & 0.193947 & 0 & 0 & 0.867466 & 0.181811 & 0 & 0 & 0.759969 \\
0.810391 & 0 & 0.835883 & 0 & 0.456507 & 0.114117 & 0 & 0.212482 & 0.746725 & 0 \\
0.189609 & 0 & 0.164117 & 0 & 0.543493 & 0.885823 & 0 & 0.787518 & 0.253275 & 0 \\
0 & 0 & 0.832375 & 0.478244 & 0.499266 & 0 & 0.842265 & 0 & 0.740099 & 0.230955 \\
0 & 0 & 0.167625 & 0.521756 & 0.500734 & 0 & 0.157735 & 0 & 0.259901 & 0.796045 \\
0 & 0.859034 & 0 & 0.478005 & 0.498000 & 0.116425 & 0 & 0.217493 & 0 & 0.203943 \\
0 & 0.140996 & 0 & 0.521995 & 0.502000 & 0.83575 & 0 & 0.782507 & 0 & 0.796057 \\
0.808242 & 0.860424 & 0 & 0.520590 & 0 & 0 & 0.843871 & 0.215010 & 0.743407 & 0 \\
0.191758 & 0.139576 & 0 & 0.479410 & 0 & 0 & 0.156129 & 0.784990 & 0.256593 & 0
\end{bmatrix}$$

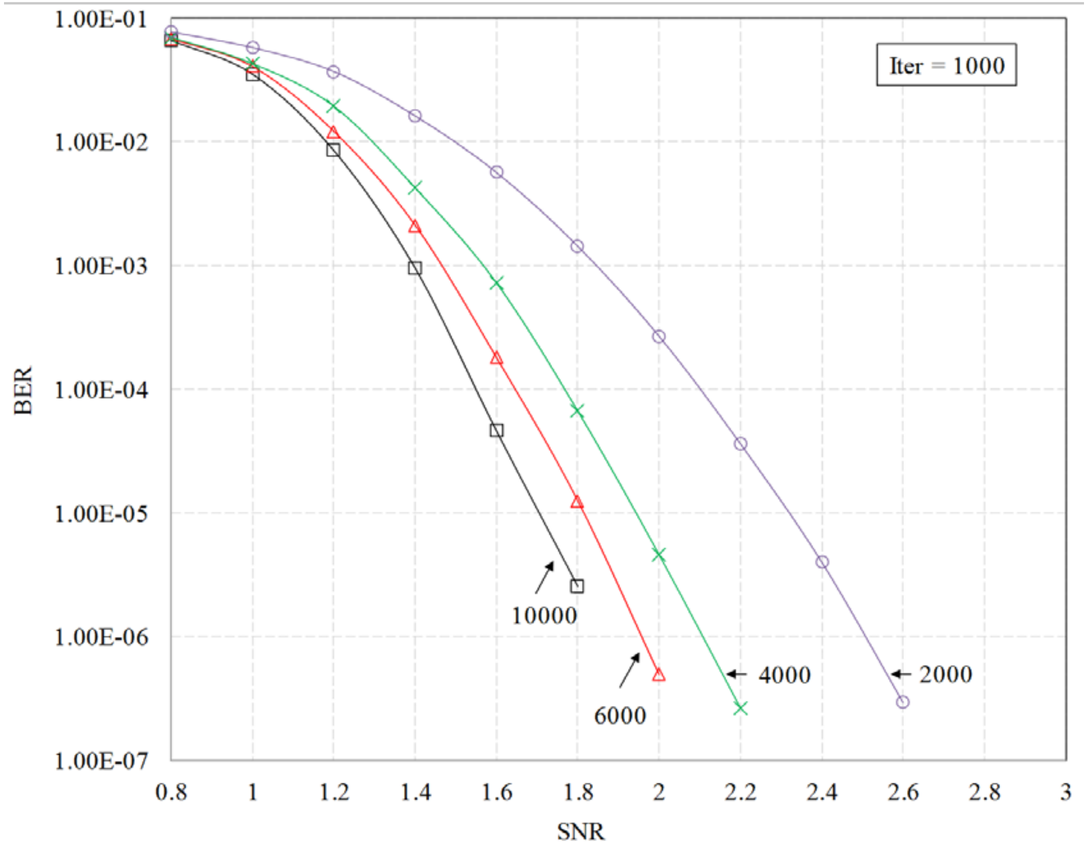
The *a posteriori* probability matrix **Q'** becomes :

$$\mathbf{Q'} = \begin{bmatrix}
0.806122 & 0.859023 & 0.832369 & 0.478419 & 0.501915 & 0.116434 & 0.842260 & 0.217514 & 0.740088 & 0.203963 \\
0.193878 & 0.140977 & 0.167631 & 0.521581 & 0.498085 & 0.883566 & 0.157740 & 0.782486 & 0.259913 & 0.796037
\end{bmatrix}$$

The estimated codeword is $\hat{c} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$. It satisfies $\hat{c} \cdot \mathbf{H}^{\mathrm{T}} = 0$ and the decoding terminates.

§ 5.4 Belief Propagation Decoding

– AWGN channel, BPSK modulation
– Design code rate: 0.5



§ 5.5 The Sum-Product Algorithm

– The BP decoding algorithm can be simplified in logrithm domain.
– Regarding bit c_n, its probabilities $\Pr(c_n = 0)$ and $\Pr(c_n = 1)$ can be unified in log likelihood ratio (LLR) as

$$LLR(c_n) = \ln \frac{\Pr(c_n = 0)}{\Pr(c_n = 1)}.$$

Inversely,

$$\Pr(c_n = 0) = \frac{1}{1 + e^{-LLR(c_n)}}, \Pr(c_n = 1) = \frac{1}{1 + e^{LLR(c_n)}}.$$

– In the Horizontal update

$$r_{mn}(\theta) = \sum_{\{c_{n'}: \Sigma\theta_{n'} = \theta\}} \prod_{n' \in N_m \backslash n} q_{mn'}(\theta).$$

If $c_n = \theta = 1$, we need to consider the permutation of $\{c_{n'}\}$ in which there are **odd** number (#) of 1s, so that $z_m = c_n + \Sigma c_{n'} = 0$. Otherwise, if $c_n = \theta = 0$, we need to consider the permutation of $\{c_{n'}\}$ in which there are **even** (#) of 1s.

§ 5.5 The Sum-Product Algorithm

– **Lemma** Given a binary sequence of length N in which each bit is independent, the probability of bit n being 1 is p_n. Then,

$$\Pr[\text{there are } \textbf{even} \text{ \# of 1s}] = \frac{1}{2} + \frac{1}{2}\prod_{n=1}^{N}(1-2p_n),$$

$$\Pr[\text{there are } \textbf{odd} \text{ \# of 1s}] = \frac{1}{2} - \frac{1}{2}\prod_{n=1}^{N}(1-2p_n).$$

– Applying the above lemma, the BP decoding becomes

Horizontal update :

$$r_{mn}(0) = \frac{1}{2} + \frac{1}{2}\prod_{n' \in N_m \backslash n}\left(1 - 2q_{mn'}(1)\right), \tag{7}$$

$$r_{mn}(1) = \frac{1}{2} - \frac{1}{2}\prod_{n' \in N_m \backslash n}\left(1 - 2q_{mn'}(1)\right). \tag{8}$$

Vertical update :

$$q_{mn}(0) = \alpha_{mn} \cdot f_n(0) \cdot \Pi_{m' \in M_n \backslash m} r_{m'n}(0), \tag{9}$$

$$q_{mn}(1) = \alpha_{mn} \cdot f_n(1) \cdot \Pi_{m' \in M_n \backslash m} r_{m'n}(1). \tag{10}$$

§ 5.5 The Sum-Product Algorithm

– Let us define the following LLR values

$$l_n = \ln\frac{f(0)}{f(1)}, u_{mn} = \ln\frac{q_{mn}(0)}{q_{mn}(1)}, v_{mn} = \ln\frac{r_{\mathrm{mn}}(0)}{r_{mn}(1)}, l_{n,p} = \ln\frac{q_n(0)}{q_n(1)}.$$

– Equip with $\tanh\frac{x}{2} = \frac{e^x-1}{e^x+1}$, $2\tanh^{-1}x = \ln\frac{1+x}{1-x}$.

– Horizontal update: $u_{mn} \to v_{mn}$

$$\tanh\frac{u_{mn}}{2} = \frac{e^{u_{mn}}-1}{e^{u_{mn}}+1} = \frac{\frac{q_{mn}(0)}{q_{mn}(1)}-1}{\frac{q_{mn}(0)}{q_{mn}(1)}+1} = \frac{q_{mn}(0)-q_{mn}(1)}{q_{mn}(0)+q_{mn}(1)} = 1-2q_{mn}(1).$$

(7) and (8) become

$$r_{mn}(0) = \frac{1}{2} + \frac{1}{2}\prod\nolimits_{n'\in N_m\backslash n}\tanh\frac{u_{mn'}}{2}, \qquad (11)$$

$$r_{mn}(1) = \frac{1}{2} - \frac{1}{2}\prod\nolimits_{n'\in N_m\backslash n}\tanh\frac{u_{mn'}}{2}. \qquad (12)$$

$$v_{mn} = \ln\frac{(11)}{(12)} = 2\tanh^{-1}\left(\prod_{n'\in N_m\backslash n}\tanh\frac{u_{mn'}}{2}\right).$$

§ 5.5 The Sum-Product Algorithm

– Vertical update : $v_{mn} \rightarrow u_{mn}$

$$u_{mn} = \ln\frac{(9)}{(10)} = \ln\frac{f(0)}{f(1)} + \sum_{m' \in M_n \backslash m} \ln\frac{r_{m'n}(0)}{r_{m'n}(1)}$$

$$u_{mn} = l_n + \sum_{m' \in M_n \backslash m} v_{m'n}$$

– Aposteriori LLR

$$l_{n,p} = \ln\frac{q_n(0)}{q_n(1)} = l_n + \sum_{m \in M_n} v_{mn}$$

– Decision on c_n

If $l_{n,p} \geq 0, \hat{c}_n = 0$; If $l_{n,p} < 0, \hat{c}_n = 1$.